THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The fourteenth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held on Saturday, December 8, 1923, in McDowell Hall, St. John's College, Annapolis, Maryland. Members of the Association were guests at lunch of the faculties of the U. S. Naval Academy, the Naval Post Graduate School, and St. John's College.

There were forty-seven (47) present including the following members of the Association:

O. S. Adams, R. N. Ashmun, G. A. Bingley, C. C. Bramble, J. A. Bullard, P. Capron, G. R. Clements, A. Cohen, L. S. Dederick, A. Dillingham, H. English, J. B. Eppes, H. W. Ficken, D. M. Garrison, W. M. Hamilton, P. E. Hemke, W. D. Lambert, E. S. Mayer, L. T. Moore, F. Morley, F. D. Murnaghan, C. A. Nelson, E. C. Phillips, C. H. Rawlins, L. J. Reed, H. M. Robert, R. E. Root, W. F. Shenton, J. Tyler, E. W. Woolard.

The following papers were presented:

(1) "Mathematics and geophysics" by Mr. W. D. LAMBERT, U. S. Coast and Geodetic Survey;

(2) "Mathematics of meteorology" by Mr. E. W. WOOLARD, U. S. Weather Bureau;

(3) "The use of mathematics in naval construction" by Commander A. J. CHANTRY, Jr., U. S. N., Head of the Department of Mathematics, U. S. Naval Academy (by invitation);

(4) "The growth function" by Dr. L. J. REED, Johns Hopkins School of Hygiene;

(5) "Mathematics of a warped airplane wing" by Professor J. B. SCARBOROUGH, U. S. Naval Academy;

(6) "Variation on an old theme" by Professor FRANK MORLEY, Johns Hopkins University;

(7) "The differential operator" by Professor JOHN TYLER, U. S. Naval Academy;

(8) "A question in the theory of numbers" by Dr. RAINICH, Johns Hopkins University (by invitation);

(9) "The minimum distance problem for four points in space" by Dr. F. D. MURNAGHAN, Johns Hopkins University.

The following are the abstracts of most of the papers: No. 1 by Mr. Lambert appeared in *Science* for Jan. 11, 1924. An abstract of it has already appeared in the MONTHLY (*1923*, 410–411). No. 9 by Dr. Murnaghan was read by title only and will appear in full in the MONTHLY.

2. The singularly fundamental rôle played by mathematics in all domains of exact scientific thought, by virtue of which any science as it becomes increasingly

mature and perfect becomes correspondingly more and more mathematical in character, was illustrated in Mr. Woolard's paper by the history and present status of the science of meteorology. Meteorology proper is the study of the mechanics and thermodynamics of the earth's atmosphere. The problem before us is: Given complete observations describing the state of an extensive three-dimensional portion of the atmosphere, to determine from the laws of mathematical physics the conditions that will result at some given time in the future. The practical difficulties in the way of obtaining adequate observational data, and the mathematical difficulties involved in the theoretical treatment of the data, are such that weather forecasting is now, and must for many generations remain, largely empirical; it is hopeless to attempt the mathematical calculation of the coming weather in the way that the movements of the heavenly bodies are forecast. Nevertheless, any insight into the mechanism of atmospheric phenomena which we may obtain through advances in theoretical meteorology will sooner or later be of practical use in improving weather predictions, and it is not unreasonable to hope that meteorology may ultimately become a truly exact science. The supreme importance of theoretical investigations into meteorological phenomena was emphasized; the past history of meteorology was compared with that of other sciences; and brief reference was made to each of the more important contributions to mathematical meteorology, including the early work of Ferrel, Helmholtz, Guldberg and Mohn, Hertz, von Bezold, Oberbeck, Sprung, Margules, *et al.*, and the modern and contemporary work of Hildebrandsson, Teisserenc de Bort, Kobayasi, Ryd, Bjerknes, Taylor, Jeffreys, Shaw, Richardson, Fujiwara, etc.

The paper is to appear in full in a future number of the *Monthly Weather Review*.

3. Commander Chantry said in substance: "In no other physical science is there found a closer relationship with pure mathematics, nor a greater and immediate direct use of it, than in naval architecture. Men have been going upon the water in ships since early history, but only in recent years have they been able to claim at least comparative mastery of the waves. The greatest advance in this mastery dates from the time that the hull of a ship was conceived as a geometric body, and was accordingly submitted to mathematical analysis.

"One of the principal steps in the design is the determination of the *lines* of the ship, which are the intersections with the hull of the ship by three series of mutually perpendicular planes. These are shown by orthographic projection on three views. This furnishes the data for finding by approximate integration the volume of the displacement of the ship up to any water line, and the center of gravity of this volume. This, known as the center of buoyancy, is of extreme importance in stability calculations, and must be found in all three dimensions. A further beautiful application of mathematics arises in the determination of the metacentric radius. In these considerations of stability, and in weight calculations, much approximate calculation of areas, volumes, centers of gravity, and moments of inertia must be made.

"In strength calculations the device of the 'equivalent girder' is employed. This brings in the whole theory of the strength of beams, and involves a great deal of detailed computation. Thus the determination at each point to be tested of I, the moment of inertia of cross-sectional area of the equivalent girder, involves the moment of inertia of members cut by a section about their own axes, and then the determination of the moment of inertia of the whole section about the neutral axis of the ship as a whole, by means of formulæ involving transfer of axes.

"In the resistance and powering of ships we find extensive and complex use of mathematics. Such a study enables us to predetermine with great accuracy the power required for a certain ship, based on the resistance of a model towed in a tank.

"A combination of experiment and mathematical investigation has given us Joessel's formula, which gives the turning moment ensuing when the rudder is put over, in terms essentially of the angle of inclination of the rudder as independent variable. The usual methods of the calculus give us the angle of inclination for maximum turning moment, and hence we need only provide for throw of rudder slightly beyond that angle.

"The behavior of a ship in a seaway is of extreme importance, especially as regards her periods of roll and pitch. These periods of roll and pitch are connected with the metacentric height of the ship by an equation derived from consideration of the ship as a pendulum, suspended from the metacenter. This equation takes the form $T = C/\sqrt{GM}$, where T is the period of roll, C is a constant, and GM is the metacentric height. From this the effect on the period of roll due to an increment in metacentric height is easily computed. Such an increment in metacentric height is most easily given by weight changes on the vessel. Thus the ship operator who knows the science of his profession may at any time alter the behavior of his ship in this respect, within certain limits.

"The anti-rolling device possessing the most promise of practicability is that involving the gyroscopic principle. Certainly in this field, no progress is possible without exhaustive and complex mathematical treatment.

"The launching of a ship involves a neat problem on the inclined plane, including determination of velocity, acceleration and coefficient of friction.

"It has been possible to touch only a few of the high spots, and such as are of basic nature. Detailed ship calculations reveal a multitude of mathematical problems. Naval architecture is indeed applied mathematics. It is hoped that two impressions may remain from this presentation. The first is a realization of the enormous contribution that mathematicians are making in the progress of naval architecture. The second is that we of our calling realize this full well and tender you our deepest appreciation of your help to us, and wish you unstinted success in further developments."

5. In 1922, Dr. Max Munk of Washington published a brief outline of a powerful and elegant method for investigating the aerodynamical behavior of a warped airplane wing of elliptic plan form. Professor Scarborough gave a

somewhat detailed exposition of the mathematical theory underlying the method.

By applying complex function theory to the fluid motion in a plane perpendicular to the direction of flight, it was shown that for a wing of elliptic form the induced and effective angles of attack have the same ratio at all points along the wing span. This gives a simple linear relation between the geometric and effective angles of attack. To get the most general warping, the geometric angle of attack is expressed as a Fourier series. Then the effective angle of attack and the density of lift along the wing span can likewise be expressed in terms of the same series.

6. The curve of pursuit for a circle leads to a differential equation which is in the ordinary sense not integrable. The point of Professor Morley's note was that, when the man M describes a circle, and the path of the dog D is always at right angles to DM, the equation is integrable. The relative path is a Cartesian oval, and the actual path can be mapped on a line explicitly.

8. Dr. Rainich gave Filippov's proof of the proposition that if the numbers $x^2 + x + m < (4m - 1)/3$ are primes all the numbers of this form which do not exceed m^2 are primes. Then with the aid of the identities

$$a^2 + ab + mb^2 = b^2(x^2 + x + m) + (a - bx)(a + bx + b)$$

for $x = 0, 1, \cdots n$ it was shown that the numbers $a^2 + ab + mb^2$, with a and b relative primes, have no prime divisors $< 2n - 1$ which do not divide at least one of the numbers $x^2 + x + m \leq n^2 + n + m$. Putting $n = m - 1$ and supposing that the numbers $x^2 + x + m < m^2$ are primes, Frobenius's theorem results that under these conditions the numbers $a^2 + ab + mb^2 < m^2$ are primes.

HARRY ENGLISH, *Secretary-Treasurer.*

DETERMINANTS WHOSE ARRAYS ARE MAGIC SQUARES.

By J. E. TREVOR, Cornell University.

1. The General Magic Array. For the present purpose, a magic square shall be understood to be an array of n^2 numbers such that the sum of the elements of each row, of each column, and of each principal diagonal is the same number s. When no further conditions are imposed, the $n^2 - 2n$ arbitrary elements may conveniently be taken to be the elements remaining when a corner element and the opposite row and column are deleted, as for example the elements x_{ij} in the array:

$$\begin{array}{lllll}
a_{11} & x_{12} & \cdots & x_{1,\,n-1} & a_{1n} \\
x_{21} & x_{22} & \cdots & x_{2,\,n-1} & a_{2n} \\
\cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot \\
x_{n-1,\,1} & x_{n-1,\,2} & \cdots & x_{n-1,\,n-1} & a_{n-1,\,n} \\
a_{n1} & a_{n2} & \cdots & a_{n,\,n-1} & a_{nn}.
\end{array}$$

Let s_b be the sum of the elements in the array B obtained by deleting the first and last rows and columns of the square; let s_d' and s_d'' be the sums of the elements of the first and second principal diagonals, respectively, of B; and let c be the sum of the four corner elements of the square. Then

$$s_b = (n-2)s - (2s - c), \qquad s_d' + s_d'' = 2s - c.$$

Hence

$$(n-2)s = s_b + s_d' + s_d'', \tag{1}$$

which expresses s in terms of the $(n-2)^2$ elements of the array B.

To complete a magic square from $n^2 - 2n$ arbitrary elements arranged as stated, it is necessary to determine a corner element. The corner elements a_{11}, a_{1n}, a_{nn} are connected by the relations

$$\begin{aligned} a_{11} + a_{1n} &= s - \sum_{j=2}^{n-1} x_{1j} \\ a_{11} + a_{nn} &= s - s_d' \qquad = s - (n-2)s + s_b + s_d'' \qquad \text{By (1).} \\ a_{1n} + a_{nn} &= s - \sum_{i=2}^{n-1} a_{in} = s - \Big[(n-2)s - \sum_{i=2}^{n-1} x_{i1} - s_b\Big]. \end{aligned}$$

On subtracting the members of the last equation from the sum of those of the first two, we find

$$2a_{11} = s + s_d'' - \sum_{i=j=2}^{n-1} (x_{1j} + x_{i1}).$$

For the summation, which is the sum of the elements in the "fringe," let us write s_f. Then

$$2a_{11} = s + s_d'' - s_f. \tag{2}$$

We observe that a_{11}, unlike s, is a function of all the arbitrary elements. The magic square determined by the $n^2 - 2n$ arbitrary elements of the above array is readily completed by employment of (1) and (2).

To the last column of the determinant D whose array is a magic square let us add all the other columns, and then to the last row add all the other rows. Hereupon, factoring s out of the last row and column, we obtain

$$D = s^2 \begin{vmatrix} a_{11} & x_{12} & \cdots & x_{1,\,n-1} & 1 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ x_{n-1,\,1} & x_{n-1,\,2} & \cdots & x_{n-1,\,n-1} & 1 \\ 1 & 1 & \cdots & 1 & n/s \end{vmatrix}.$$

Let us write A for the determinant obtained by deleting the last row and column of D, and A_i' for the determinant obtained from A by replacing each element of its ith row by unity. Then, on expanding the above determinant with reference to the elements of its last column, the equation becomes

$$D = s^2\Big(\frac{n}{s}A - \sum_{i=1}^{n-1} A_i'\Big). \tag{3}$$

Let $\Delta^{(k)}$ denote the determinant formed from any given determinant Δ by adding an arbitrary constant k to each element of Δ. Then

$$A^{(k)} = A + k\sum_{i=1}^{n-1} A_i', \qquad \partial A^{(k)}/\partial k = \sum_{i=1}^{n-1} A_i'.$$

Hence (3) becomes

$$\frac{D}{s} = nA - s\frac{\partial A^{(k)}}{\partial k}. \tag{4}$$

In seeking to determine the effect of the addition of the arbitrary constant k to each independent variable in D, we observe, by (1) and (2), that such addition increases s by nk, and increases each dependent element of the array of D by k. And we note that the value $\sum A_i'$ of $\partial A^{(k)}/\partial k$ is unaffected. So, for the determinant $D^{(k)}$, the equation (4) is

$$\frac{D^{(k)}}{s+nk} = nA^{(k)} - (s+nk)\frac{\partial A^{(k)}}{\partial k}; \tag{5}$$

whence, differentiating,

$$\frac{\partial}{\partial k}\frac{D^{(k)}}{s+nk} = 0; \tag{5a}$$

i.e., the ratio differentiated is independent of k. This result establishes the following theorem.

THEOREM. *When D is a determinant, of order n, whose array is a magic square having the row-sum s, the ratio D/s is a function of the differences of the $n^2 - 2n$ independent variables x_{ij} that determine D.*

2. Odd-rowed Concentric Magic Arrays. Consider any odd-rowed magic square from which successive magic squares are obtained by successive removal of the bounding rows and columns. Let x_1 be the central element. Then the arbitrary fringe-elements X_{31}, x_{31} determine the corner element c_3 and the other dependent elements of the next square, of order three. Continued repetition of the process yields the following array, in which x_1 and the X_{ij} and x_{ij} are the $(n^2 - 2n + 3)$ arbitrary elements.

$$\begin{array}{lllllll}
c_n & x_{n1} & x_{n2} & \cdot & \cdot & x_{n,\,n-2} & c_{n1} \\
X_{n,\,n-2} & \cdot & \cdot & \cdot & \cdot & \cdot & A_{n,\,n-2} \\
\cdot & \cdot & c_3 & x_{31} & c_{31} & \cdot & \cdot \\
\cdot & \cdot & X_{31} & x_1 & A_{31} & \cdot & \cdot \\
X_{n2} & \cdot & c_{32} & a_{31} & c_{33} & \cdot & A_{n2} \\
X_{n1} & \cdot & \cdot & \cdot & \cdot & \cdot & A_{n1} \\
c_{n2} & a_{n1} & a_{n2} & \cdot & \cdot & a_{n,\,n-2} & c_{n3}
\end{array}$$

By (1), $(n-2)s_n = s_b + s_d' + s_d''$, the row-sum s_n for the array of the nth order is given by $(n-2)s_n = (n-2)s_{n-2} + 2s_{n-2}$. Hence we have, successively,

$$s_n = \frac{n}{n-2}s_{n-2}, \qquad s_{n-2} = \frac{n-2}{n-4}s_{n-4}, \qquad \cdots, \qquad s_5 = \frac{5}{3}s_3, \qquad s_3 = \frac{3}{1}x_1;$$

wherefore

$$s_n = nx_1. \tag{6}$$

By (2), $2a_{11} = s + s_d'' - s_f$, the corner elements c_i are given by $2c_n = s_n + s_{n-2} - F_n - f_n$, if we write F_n, f_n for the sums of the fringe-elements X_{nj}, x_{nj} of the n-rowed square. Substituting (6),

$$c_n = (n-1)x_1 - \frac{1}{2}(F_n + f_n). \tag{6a}$$

Further, since the sum of the elements of any column or row terminating in a_{nj} or A_{nj} is equal to s_n, we have $x_{nj} + s_{n-2} + a_{nj} = s_n$, and $X_{nj} + s_{n-2} + A_{nj} = s_n$, where $s_n = nx_1$, and $s_{n-2} = (n-2)x_1$. Hence

$$a_{nj} = 2x_1 - x_{nj}, \qquad A_{nj} = 2x_1 - X_{nj}. \tag{6b}$$

Finally, if we form $s_n = nx_1$ by summation of the elements of the first row, of the first column, and of the first diagonal,

$$c_n + f_n + c_{n1} = nx_1, \qquad c_n + F_n + c_{n2} = nx_1, \qquad c_n + (n-2)x_1 + c_{n3} = nx_1,$$

the relation (6a) reduces these to

$$\begin{gathered} c_{n1} = x_1 + \tfrac{1}{2}(F_n - f_n), \qquad c_{n2} = x_1 - \tfrac{1}{2}(F_n - f_n), \\ c_{n3} = -(n-3)x_1 + \frac{1}{2}(F_n + f_n). \end{gathered} \tag{6c}$$

Let D be the determinant whose array is that under examination, and let A_n be its minor A. By (5) and (5a) we have

$$\frac{D}{s_n} = \frac{D^{(k)}}{s_n + nk} = nA_n{}^{(k)} - (s_n + nk)\frac{\partial A_n{}^{(k)}}{\partial k}.$$

To exhibit the fact that D/s_n is a function of differences of the variables, put $k = -x_1$. Then[1]

$$s_n = nx_1, \qquad nk = -nx_1, \qquad s_n + nk = 0;$$

[1] For $k = -x_1$, the initial equation above is

$$D/s_n = D^{(-x_1)}/0.$$

If $x_1 = 0$, then $s_n = nx_1 = 0$, and hence $D = 0$, since D contains s_n as a factor. *I.e.*, a determinant whose array is an odd-rowed concentric magic square whose central element is zero vanishes identically.

This result follows from the curious symmetry of $D^{(-s_1)}$. For we have

$$c_{n3} - x_1 = -(c_n - x_1), \qquad c_{n2} - x_1 = -(c_{n1} - x_1), \qquad A_{nj} - x_1 = -(X_{nj} - x_1), \qquad a_{nj} - x_1 = -(x_{nj} - x_1),$$

and the central element is zero. It thus appears that the elements on the principal diagonals have center-symmetry with change of sign, while the elements $X_{nj} - x_1$ (or $x_{nj} - x_1$) are equal to the negatives of the opposite elements $A_{nj} - x_1$ (or $a_{nj} - x_1$) in the same row (or column). Any determinant of this form vanishes identically.

The general determinant having the above-described type of symmetry has an interesting expansion. When the central element y_1 is arbitrary, the formation of the odd-rowed array is

and the above equation becomes

$$D/s_n = nA_n^{(-x_1)}. \tag{7}$$

We now proceed to replace the dependent variables in A_n by their values from (6a), (6b), (6c), and to form the determinant $A_n^{(-x_1)}$ by adding $-x_1$ to each element of A_n. To reduce the determinant so obtained we first add to the last row all the other rows except the first, whereby the last row becomes

$$\sum_{j=1}^{n-2}(X_{nj} - x_1) \quad 0 \quad 0 \quad \cdots \quad 0.$$

Hence, if C is the cofactor of the non-vanishing element of this row, the determinant is equal to $-\sum (X_{nj} - x_1)\cdot C$, the product being negative because $n-1$ is even. Next, in C, we add to the last column all the other columns, whereby the last column becomes

$$\sum_{j=1}^{n-2}(x_{nj} - x_1) \quad 0 \quad 0 \quad \cdots \quad 0.$$

So C is equal to $+\sum (x_{nj} - x_1)\cdot A_{n-2}^{(-x_1)}$, the product being positive because $n-2$ is odd. Hence

$$A_n^{(-x_1)} = -\sum_{j=1}^{n-2}(X_{nj} - x_1)\cdot \sum_{j=1}^{n-2}(x_{nj} - x_1)\cdot A_{n-2}^{(-x_1)}.$$

Thus each application of the foregoing process reverses the sign of the expression obtained. Continued repetition of the process yields a succession of similar

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illustrated by the case $n = 5$, which may be written as follows:

$$\Delta_5 = \begin{vmatrix} Y_5 & x_{51} & x_{52} & x_{53} & y_5 \\ X_{53} & Y_3 & x_{31} & y_3 & -X_{53} \\ X_{52} & X_{31} & y_1 & -X_{31} & -X_{52} \\ X_{51} & -y_3 & -x_{31} & -Y_3 & -X_{51} \\ -y_5 & -x_{51} & -x_{52} & -x_{53} & -Y_5 \end{vmatrix}.$$

In reducing this determinant, to the last row add the first row, and to the last column add the first column. Expansion then gives

$$\Delta_5 = -(Y_5^2 - y_5^2)\Delta_3.$$

On beginning with the determinant Δ_n of order n, continued repetition of this operation yields

$$\Delta_n = (-1)^{(n-1)/2}\cdot y_1 \cdot \prod_{n=3}^{n}(Y_n^2 - y_n^2), \quad (n \text{ odd}),$$

which is the expansion in question.

When n is even, the two-rowed kernel of the array is

$$\begin{matrix} Y_2 & y_2 \\ -y_2 & -Y_2 \end{matrix}$$

and the expansion is

$$\Delta_n = (-1)^{n/2}\cdot \prod_{n=2}^{n}(Y_n^2 - y_n^2), \quad (n \text{ even}).$$

factors,

$$A_n^{(-x_1)} = (-1)^{(n-1)/2} \prod_{n=3}^{n} \left(\sum_{j=1}^{n-2} (X_{nj} - x_1) \cdot \sum_{j=1}^{n-2} (x_{nj} - x_1) \right) \cdot$$

On substituting this value in (7), which is $D = ns_n \cdot A_n^{(-x_1)} = n^2 x_1 A_n^{(-x_1)}$, we are enabled to state the following theorem.

THEOREM. *When D is a determinant, of order n, whose array is an odd-rowed concentric magic square having the row-sum s_n, the ratio D/ns_n or D/n^2x_1 is a product of linear factors such that*

$$D = (-1)^{(n-1)/2} \cdot n^2 x_1 \cdot \prod_{n=3}^{n} \left(\sum_{j=1}^{n-2} (X_{nj} - x_1) \cdot \sum_{j=1}^{n-2} (x_{nj} - x_1) \right), \quad (n \text{ odd}).$$

It should be noted that every magic square of order three is a concentric magic square, and hence that every determinant whose array is a three-rowed magic has the form

$$-3^2 x_1 (X_{31} - x_1)(x_{31} - x_1).$$

3. Even-rowed Concentric Magic Arrays. We shall consider three cases of even-rowed magic squares from which successive magic squares are obtained by successive removal of the bounding rows and columns. If the two-rowed kernel of the square is to be magic, it will be an array of identical elements; if the kernel is freed from the summation-condition on its diagonals, each diagonal will consist of identical elements; if the kernel is freed from all conditions, it will be an array of arbitrary elements. In these successive cases the kernels are

$$\begin{array}{lll} x_1x_1 & x_1x_2 & x_1x_2 \\ x_1x_1 & x_2x_1 & x_3x_4 \end{array}$$

In the first case, let the first and second rows of the kernel be x_2, a_2 and a_2, a_2 respectively.[1] The arbitrary fringe-elements X_{41}, X_{42}, x_{41}, x_{42} determine the corner element c_4 and the other dependent elements of the next square, of order four. Continued repetition of the process yields an array similar to that displayed in the preceding section, and in which x_2 and the X_{ij} and x_{ij} are the $(n^2 - 2n + 2)/2$ arbitrary elements. The process of deducing (6), $\cdots$, (6c) yields here the same equations, save that x_1 is everywhere replaced by x_2; and we again find $D/s_n = nA_n^{(-x_2)}$. On evaluating $A_n^{(-x_2)}$ as before, we find it to reduce to a product as in the preceding case, save that the last factor is

$$\begin{vmatrix} c_4 - x_2 & x_{41} - x_2 & x_{42} - x_2 \\ X_{42} - x_2 & 0 & 0 \\ X_{41} - x_2 & 0 & 0 \end{vmatrix} = 0.$$

In this case, then, D vanishes identically.

In the second case we build up the concentric magics as before about the kernel $|x_1\ x_2,\ x_2\ x_1|$, obtaining an array in which x_1, x_2 and the X_{ij} and the x_{ij}

[1] Here, of course, it is meant that $a_2 = x_2$.

are the $(n^2 - 2n + 4)/2$ arbitrary elements. The process of deducing (6), $\cdots$, (6c) yields here the same equations for the dependent elements, save that x_1 is everywhere replaced by $(x_1 + x_2)/2$, and that the corner elements of the four-rowed square are

$$\begin{array}{l|l} c_4 = x_1 + 2x_2 - \frac{1}{2}(F_4 + f_4) & c_{42} = x_1 - \frac{1}{2}(F_4 - f_4) \\ c_{41} = x_1 + \frac{1}{2}(F_4 - f_4) & c_{43} = -x_1 + \frac{1}{2}(F_4 + f_4). \end{array}$$

In deducing the equation analogous to (6), we choose $k = -(x_1 + x_2)/2 = -\xi$, and so obtain

$$D/s_n = nA_n^{(-\xi)}.$$

On evaluating $A_n^{(-\xi)}$ in the same way as before, we find

$$A_n^{(-\xi)} = (-1)^{(n-2)/2} \cdot \frac{x_2 - x_1}{2} \cdot \prod_{n=4}^{n} \left(\sum_{j=1}^{n-2} (X_{nj} - \xi) \cdot \sum_{j=1}^{n-2} (x_{nj} - \xi) \right).$$

On substituting this value in the above equation,

$$D = ns_n A_n^{(-\xi)} = n^2 \frac{x_1 + x_2}{2} A_n^{(-\xi)},$$

we are enabled to state the following theorem.

THEOREM. *When D is a determinant, of order n, whose array is an even-rowed concentric magic square having a two-rowed kernel of identical elements and the row-sum s, the value of D is zero. But when the kernel has the form $|x_1\, x_2,\ x_2\, x_1|$, the ratio D/ns_n or $2D/n^2(x_1 + x_2)$ is a product of linear factors such that*

$$D = (-1)^{(n-2)/2} \cdot n^2 \frac{x_2^2 - x_1^2}{4} \cdot \prod_{n=4}^{n} \left(\sum_{j=1}^{n-2} (X_{nj} - \xi) \cdot \sum_{j=1}^{n-2} (x_{nj} - \xi) \right), \qquad (n \text{ even}),$$

where $\xi = (x_1 + x_2)/2$.

In the third case cited, in which the array of the determinant is a concentric magic square having a kernel of arbitrary elements, the procedure employed here leads to no simple formulation.

THE COCHLIOID.[1]

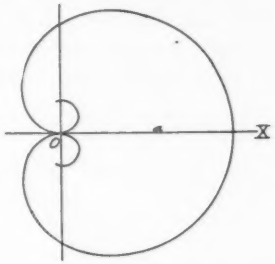
By ROSCOE WOODS, University of Iowa.

1. Introduction. The term "cochlioid" as applied to a curve is now used in a different sense from that employed by the ancients. It was the name given by Pappus to Nicomedes' conchoid which was used as a means of solving the famous problems of trisecting an angle and duplicating a cube. Also Apollonius of Perga called the quadratix of Dionostratus the sister of the cochlioid.[2] But in more recent times Benthem and Falkenburg[3] associated the name "Cochleoïde"

[1] Read before the Iowa Section of the Mathematical Association of America, April 28, 1923.
[2] P. Tannery, *Bulletin des Sciences Mathématiques*, 1883, p. 283.
[3] *Nieuw Archief voor Wiskunde*, t. X, 1883, p. 76.

with the curve whose equation in polar coördinates is $r\theta = a \sin \theta$, and Benthem notes that Catalan gave this equation[1] in 1857. In the *Philosophical Transactions of the Royal Society of London*, 1706, there is an anonymous discussion[2] of the equation $r(\pi - 2\omega) = 4a \cos \omega$. E. Wölffing has shown that this article was by J. Perks.

In the present article, the curve whose equation in polar coördinates is $r\theta = a \sin \theta$ will be designated as the cochlioid.[3] My interest in the curve began when I obtained it as the trace of a helical beam of electrons on a plane perpendicular to its axis. The data, described later in the paper, was furnished by Dr. C. J. Lapp. In what follows detailed discussion is given only in case of special interest. The graph and some of the properties of the curve are first exhibited. Then follows five methods of defining the curve. Finally there are two paragraphs on the tangents and normals.



2. Graph. The graph of the cochlioid is easily obtained from its equation. The curve is symmetrical to the polar axis. In each half of the plane we find an infinite number of non-intersecting ovals, each of which is tangent to the polar axis at the pole, O. The intercept on the polar axis is a. At this point the radius of curvature is $3a/4$. The intercepts (taken without regard to sign) on any radius vector except the polar axis form a harmonic divergent series. If the radius vector makes an angle φ with the polar axis, the intercepts are

$$\frac{a \sin \varphi}{\varphi}, \quad -\frac{a \sin \varphi}{\varphi + \pi}, \quad \frac{a \sin \varphi}{\varphi + 2\pi}, \quad \ldots.$$

The area swept over by the radius vector is

$$\frac{1}{2}\int_0^\theta r^2 d\theta = \frac{a^2}{2}\int_0^\theta \frac{\sin^2 \theta}{\theta^2}\, d\theta.$$

This integral cannot be expressed in terms of the elementary functions, but if θ varies from 0 to ∞, the limiting value of the area is $\pi a^2/4$,[4] or one fourth the area of a circle of radius a.

3. Methods of defining the cochlioid. There are several ways of defining the cochlioid. Some of the most important are as follows:

(a) The cochlioid is the locus of the center of gravity of a variable arc measured from a fixed point X on a circle of radius $OX = a$.[5]

[1] E. Catalan, *Manuel des Candidats à l'Ecole Polytechnique*, vol. 1, 1857, p. 331.

[2] Gino Loria, *Spezielle Algebraische und Transcendente Ebene Kurven*, 1902, pp. 418–424.

[3] It may be of interest to point out that this equation enjoys the distinction of having been given by the Oui-ja board. See Oliver Lodge, *The Survival of Man*, pp. 130-134.

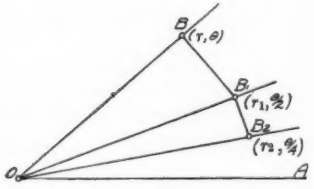
[4] Byerly, *Integral Calculus*, p. 105.

[5] E. Egger, *Ann. di Matem.* (1864), p. 21. Also L. Stoeckley, *Archiv der Math. u. Physik*, first series, 1868, p. 110. See also Haton de la Goupillière, *Comptes Rendus de l'Académie des Sciences*, 1906, p. 1130.

(*b*) If upon all the circles tangent to the x-axis at the origin O an arc of constant length is measured from O, the locus of the end point of this arc is the cochlioid.[1] Or, as Falkenburg has stated it:[2] Consider a variable isosceles triangle OVP whose legs are OV and PV. The vertex V moves on the Y-axis. O is the origin so that the leg OV coincides with the Y-axis. If PV makes an angle with the Y-axis that varies inversely as the distance OV, then the vertex P describes the cochlioid.

(*c*) In 1874, Gregor Fontana[3] proposed and solved this problem. Draw a line OB making an angle θ with a fixed line OA. At any point B on OB erect a perpendicular and let it cut the bisector of the angle θ in B_1. At B_1 erect a perpendicular and let it cut the bisector of $\theta/2$ in B_2, and so on. Continuing this process indefinitely, what is the limiting position of B_i on OA and upon what curve will the points B_i lie?

Let the coördinates of B be (r, θ) and those of B_i be $(r_i, \theta/2^i)$ for $i = 1, 2, 3, \cdots$. From the figure we then have $r_1 = r/\cos(\theta/2), \cdots, r_n = r_{n-1}/\cos(\theta/2^n)$. Hence $r_n = r/(\prod_1^n \cos\theta/2^i)$ and since we can write $\sin\theta = 2^n \sin(\theta/2^n)\prod_1^n \cos(\theta/2^i)$ we have, after rearranging the terms and substituting in the above, $r_\infty \sin\theta = r\theta$ for n approaching infinity. If the curve is required to pass through the point (a, φ), it then has the form $r\theta \sin\varphi = a\varphi \sin\theta$. If we let φ approach zero, we have the cochlioid in its simple form.



(*d*) The cochlioid appears as the solution of the following differential equation, $(x^2 + y^2)(xdy - ydx) - a(x^2 - y^2)dy + 2axydx = 0$. This equation is a special type of a more general differential equation studied by G. Fouret.[4] If a point on a right helicoid surface is luminous, Fouret showed that the projection, on a plane perpendicular to the axis of the helicoid, of the boundary of the shadow is a cochlioid. From this Brocard deduced that the cochlioid is the projection of a circular helix, from a point on the same, on a plane perpendicular to its axis.

(*e*) The trace of a helical beam of electrons on a plane perpendicular to its axis is the cochlioid.

If a fine stream of electrons is caught in a magnetic field, the paths of the electrons become helices on cylinders of radius r, where $r = mV_v/eH$, in which m is the mass of the electron, V_v its radial velocity, e its charge and H the strength of the magnetic field. If the velocity of the stream is uniform and equal to V and if the beam is projected at an angle φ with the horizontal, we have the following relations, $V_v = V \sin\varphi$ and $V_h = V \cos\varphi$. If a plate p, at a distance S

[1] A problem considered by Bernoulli and Goldbach in 1726.

[2] *Archiv der Math. u. Physik*, first series, 70, p. 257.

[3] *Mem. de la Soc. della Scienze*, 11 (1874).

[4] *Bulletin de la Société Mathématique*, t. 7, 1879, p. 199. See also H. Brocard, *Mathesis*, 1901, p. 109.

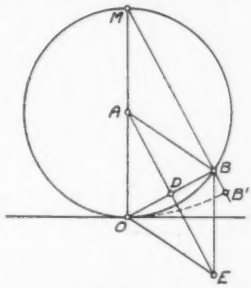
from the source, be inserted across the path of the helix, perpendicular to its axis, the coördinates of the position of the electron in p are (with the proper choice of the X-axis)

$$X = S, \qquad Y = r \sin \theta, \qquad Z = r(1 - \cos \theta), \tag{1}$$

where θ is the angle measured about the axis of the cylinder about which the electron rotates during its flight. Further we know that

$$T = S/V_h = (n + \lambda)t, \tag{2}$$

where T is the time required for the electron to traverse the distance S, t is the time of one convolution (being constant for electrons of any velocity in a given field) and $n + \lambda$ is the number of convolutions of the electron during its flight in which n is an integer and λ is a fraction. Since $\theta = (n + \lambda)2\pi$, θ and r can be eliminated from equations (1). If this result be thrown into polar coördinates, the equation of the trace of the electrons in p has the form $r = k \sin \pi\lambda/(n + \lambda)\pi$, where k is a constant.[1]



In this connection, it might be well to point out that the cochlioid is the inverse of the quadratix, $y = x \cot \pi x/2a$. It can therefore be used to square the circle. By means of Falkenburg's interpretation of Goldbach's problem, it can be used to divide an angle into any given number of equal parts. In fact Falkenburg discovered the cochlioid in connection with the design of the starting gear of a steam engine. In the adjacent figure, let $OA = r$, angle $OAB = \pi a/r$, E be the fourth vertex of the rhombus $OEBA$, and let D be the intersection of its diagonals. Then it can be proved that B and B' describe the same cochlioid if $MB' = MO$. The point E describes a hyperbolic spiral and D a second cochlioid. The variable line AB envelops a curve E' whose equation is[2]

$$\sin\left(\frac{ay + \sqrt{ax(x^2 - y^2 - ax)}}{x^2 + y^2}\right) = \frac{y\sqrt{ax} - x\sqrt{x^2 - y^2 - ax}}{x^2 + y^2}. \tag{3}$$

Furthermore the cochlioid is the pedal of E' with respect to O.

4. Tangents. The equation of the tangent at the point (r_1, θ_1) is

$$ar \sin(\theta - 2\theta_1) + ar_1 \sin \theta_1 - rr_1 \sin(\theta - \theta_1) = 0. \tag{4}$$

This equation is always satisfied by the point $(a, 2\theta_1)$. Therefore all the tangents to the cochlioid at its intersections with a given radius vector from the pole, O, pass through the same point, the locus of which is a circle with center O and radius

[1] The experimental work of this problem was carried on by Dr. C. J. Lapp while at the University of Illinois. Dr. Lapp is now connected with the Physics Department of the State University of Iowa. He has photographs of the traces which serve as a partial check on the above equation. Since I found the above equation in this connection, I was led to investigate the curve. See *Transactions of Illinois Academy of Science*, vol. 15, p. 331.

[2] Neuberg, *Wiskundige Opgaven*, vol. 2, 1884, p. 249; also *Mathesis*, 1885, p. 91.

a. This theorem is due to Cesàro who also noted that the tangent to the curve generated by the center of gravity of a variable arc always passes through the moving point on the fixed circle.[1]

The points of contact of a system of parallel tangents lie on

$$r = a \sin (2\theta - \varphi)/\sin (\theta - \varphi), \tag{5}$$

where φ is the angle between the tangents and the polar axis. This curve is an oblique strophoid. If $\varphi = \pi/2$, it becomes the right strophoid, but it degenerates into a circle if φ becomes zero.

5. Normals. For a discussion of the normals it is advisable to throw the equation of the curve into rectangular coördinates. It is

$$ay = (x^2 + y^2) \arctan y/x. \tag{6}$$

The equation of the normal at the point (x, y) is

$$(x^2 + y^2)^2 - (x^2 + y^2)(xX + yY + ax) + aX(x^2 - y^2) + 2axyY = 0, \tag{7}$$

where X, Y are running coördinates. But if we think of the point (X, Y) as fixed, the feet of the normals from the point (X, Y) to the cochlioid lie on the bicircular quartic (7).

If the equation (7) is required to be the product of the equations of two circles, the point (X, Y) must lie on the strophoid whose equation is

$$Y^2 = X(X - a)^2/(2a - X). \tag{8}$$

The vertex of this curve is at the origin, its double point is $(a, 0)$ and its asymptote is the line $X = 2a$. This curve can be represented parametrically by the equations

$$X = 2at^2/(1 + t^2), \qquad Y = at(t^2 - 1)/(1 + t^2). \tag{9}$$

If we substitute these values in (7) and factor, the equations of the two circles are found to be

$$x^2 + y^2 - ax - aty = 0 \quad \text{and} \quad (1 + t^2)(x^2 + y^2) - 2at^2x + 2aty = 0, \tag{10}$$

where t can have all real values. The centers of these two circles generate two curves when t takes all its values. These curves are

$$x^2 + y^2 - ax = 0 \quad \text{and} \quad x = a/2. \tag{11}$$

If we eliminate t from equations (10), the equation of the variable intersection of these two circles is found to be

$$(x^2 + y^2 - ax)^2 + 2a(a - x)(x^2 + y^2 - ax) + a^2y^2 = 0. \tag{12}$$

The x-intercepts of this curve are 0, a, and $2a$. The point $(a, 0)$ is a double point of this curve which in shape is very similar to the figure eight.

[1] *Nouvelle Correspondance Mathématique*, t. IV (1878), p. 283.

Keeping X, Y constant, the quartic (7) is the inverse of the conic K, where

$$K \equiv aXx^2 - aXy^2 + 2axyY - ax - xX - yY + 1 = 0. \tag{13}$$

The condition that K be two straight lines is (8). If X, Y runs over the strophoid (8), the intersection of the two lines that form K generates the strophoid whose equation is

$$a^2Y^2 = (1 - aX)^2(2aX - 1)/(3 - 2aX). \tag{14}$$

Finally if in the quartic (7) we set $y = \pm ix - u$, where $i = \sqrt{-1}$ and let x approach infinity in the result, the following quadratic equation is obtained,

$$2u^2 + (Y \mp iX \mp ia)u - a(X \pm iY) = 0. \tag{15}$$

If the roots of this quadratic are unequal, the quartic (7) has four distinct asymptotes, two passing through each circular point. Hence the quartic (7) has a node at each circular point. If the roots are equal, the asymptotes coincide and the quartic (7) has a cusp at each circular point. The condition that (15) has equal roots is

$$(Y \mp iX \mp ia)^2 + 8a(X \pm iY) = 0 \tag{16}$$

or

$$Y^2 - X^2 + 6aX - a^2 = 0 \qquad \text{and} \qquad Y(X + a \pm 4a) = 0. \tag{17}$$

For the plus sign the points of intersection are $[a(3 \pm 2\sqrt{2}),\ 0]$ and $(-5a \pm 2a\sqrt{14})$. For the minus sign, they are $[a(3 \pm 2\sqrt{2}),\ 0]$ and $(3a, \pm 2ai\sqrt{2})$. If X, Y are restricted to real quantities, there are only four positions in the plane for the point (X, Y) which make the quartic (7) have two cusps on the line at infinity. When the condition (16) is satisfied, the quartic (7) is known as a "cartesian."

THE CORRELATION BETWEEN TWO VARIATES ONE OF WHICH IS NORMALLY DISTRIBUTED.[1]

By P. R. RIDER, Washington University.

It is the purpose of this paper to discover the correlation between two variates x and y ($y = kx^n$, $k > 0$), where x is distributed according to the so-called normal law of error. A problem of this type would arise if, for instance, one wished to find the correlation between the diameters and the weights of a set of homogeneous spheres, either the diameters or the weights being normally distributed. A concrete example might be afforded by the apples on a tree. Professor Rietz[2] has given other practical illustrations in discussing the frequency distribution

[1] Presented before the American Mathematical Society, April 19, 1924.

[2] H. L. Rietz, "Frequency distributions obtained by certain transformations of normally distributed variates," *Annals of Math.* (2), vol. 23, pp. 292–300.

of the second variate and has emphasized the importance of considering the properties of the frequency curve of this variate, which is obtained from the variate x by means of the transformation given above, *viz.*, $y = kx^n$, $k > 0$. It would consequently seem worth while to determine the correlation between these two variates, even though they do not vary independently.

Let us assume that the variate x is measured in terms of its standard deviation as a unit, and that its frequency is given by $\varphi(x - \bar{x})$, where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}.$$

The coefficient of correlation, which is independent of k, is then given by the formula [1]

$$r = \frac{\int_0^{2\bar{x}} (x - \bar{x})(x^n - \bar{y})\varphi(x - \bar{x})dx}{\left[\int_0^{2\bar{x}} (x - \bar{x})^2\varphi(x - \bar{x})dx\right]^{1/2}\left[\int_0^{2\bar{x}} (x^n - \bar{y})^2\varphi(x - \bar{x})dx\right]^{1/2}}, \tag{1}$$

in which $\bar{y}$ is the mean of the variate y, *i.e.*,

$$\bar{y} = \frac{\int_0^{2\bar{x}} x^n\varphi(x - \bar{x})dx}{\int_0^{2\bar{x}} \varphi(x - \bar{x})dx}. \tag{2}$$

It will be noted that the limits of the integrals involved in r have been taken as 0 and $2\bar{x}$, it being assumed for convenience that $\bar{x} > 0$. This avoids negative values of x, which lead to imaginary values of certain of the functions dealt with, and simplifies some of the formulas subsequently developed. Moreover, unless limits which are symmetric with respect to $\bar{x}$ are taken, it is not identically true that

$$\bar{x} = \frac{\int_0^{2\bar{x}} x\varphi(x - \bar{x})dx}{\int_0^{2\bar{x}} \varphi(x - \bar{x})dx}. \tag{3}$$

Because of the rapid approach of the function $\varphi(x - \bar{x})$ to zero with increasing absolute value of $x - \bar{x}$, it will not seriously affect the value of r to cut off the integrals at these limits provided $\bar{x}$ is moderately large and n moderately small (say $\bar{x} > 4$, $n < 10$). We shall for the present restrict ourselves to values of n greater than $-\frac{1}{2}$, in which case the integrals contained in r will all be finite, even though some of them may be improper.

[1] Cf. E. V. Huntington, "Mathematics and statistics, with an elementary account of the correlation coefficient and the correlation ratio," this MONTHLY (*1919*, 421–435).

It follows quite readily from (1), if we make use of (2) and (3), that

$$r = \frac{\int_0^{2\bar{x}} (x^{n+1} - \bar{x}\bar{y})\varphi(x - \bar{x})dx}{\left[\int_0^{2\bar{x}} (x^2 - \bar{x}^2)\varphi(x - \bar{x})dx\right]^{1/2}\left[\int_0^{2\bar{x}} (x^{2n} - \bar{y}^2)\varphi(x - \bar{x})dx\right]^{1/2}}$$

$$= \frac{\int_0^{2\bar{x}} x^n(x - \bar{x})\varphi(x - \bar{x})dx}{\left[\int_0^{2\bar{x}} (x^2 - \bar{x}^2)\varphi(x - \bar{x})dx\right]^{1/2}\left[\int_0^{2\bar{x}} x^{2n}\varphi(x - \bar{x})dx - \frac{\left\{\int_0^{2\bar{x}} x^n\varphi(x - \bar{x})dx\right\}^2}{\int_0^{2\bar{x}} \varphi(x - \bar{x})dx}\right]^{1/2}} \quad (4)$$

$$= \frac{\int_0^{2\bar{x}} x^n(x - \bar{x})\varphi(x - \bar{x})dx}{[\alpha(\bar{x}) - 2\bar{x}\varphi(\bar{x})]^{1/2}\left[\int_0^{2\bar{x}} x^{2n}\varphi(x - \bar{x})dx - \frac{1}{\alpha(\bar{x})}\left\{\int_0^{2\bar{x}} x^n\varphi(x - \bar{x})dx\right\}^2\right]^{1/2}} \quad (5)$$

where

$$\alpha(x) = \int_{-x}^{x} \varphi(x)dx.$$

The first factor in the denominator of (5) is approximately equal to unity when $\bar{x}$ is moderately large. For example, if $\bar{x} = 4$, we have $\alpha(4) = 0.99994$, $\varphi(4) = 0.00013$, $[\alpha(4) - 2 \times 4\varphi(4)]^{1/2} = 0.9994$.

If in (1) we let $f(x) = (x - \bar{x})\sqrt{\varphi(x - \bar{x})}$, $g(x) = (x^n - \bar{y})\sqrt{\varphi(x - \bar{x})}$, we find that

$$r = \frac{\int_0^{2\bar{x}} f(x)g(x)dx}{\left[\int_0^{2\bar{x}} f^2(x)dx\right]^{1/2}\left[\int_0^{2\bar{x}} g^2(x)dx\right]^{1/2}},$$

and it follows from Schwarz's inequality [1] that the numerator of r is not greater in absolute value than the denominator. Thus the maximum absolute value that r can attain is unity. It can be shown directly that $r = 1$ when $n = 1$, as is to be expected, since perfect correlation exists between two variates when and only when each is a linear function of the other.

In the expressions (4) and (5) the numerator is $\int_0^{2\bar{x}} x^n(x - \bar{x})\varphi(x - \bar{x})dx$. Now the function $(x - \bar{x})\varphi(x - \bar{x})$ appearing in the integrand is symmetric with respect to the point $(\bar{x}, 0)$, and since x^n increases or decreases when x increases from 0 to $2\bar{x}$ according as n is positive or negative, it follows that

$$\int_{\bar{x}}^{2\bar{x}} x^n(x - \bar{x})\varphi(x - \bar{x})dx \gtrless \int_0^{\bar{x}} x^n(x - \bar{x})\varphi(x - \bar{x})dx,$$

[1] See G. Vivanti, *Elementi della Teoria delle Equazioni Integrali Lineari*, p. 213.

according as $n \gtrless 0$, and therefore that

$$\int_0^{2\bar{x}} x^n(x-\bar{x})\varphi(x-\bar{x})dx \gtrless 0,$$

according as $n \gtrless 0$. Thus, since the denominator of r is positive, r is positive for positive values of n and negative for negative values of n. It can be shown by a direct evaluation that $r = 0$ when $n = 0$.

Let us now determine the limiting value of r as n approaches infinity. By the law of the mean for definite integrals, we have

$$\int_0^{2\bar{x}} x^m\varphi(x-\bar{x})dx = \varphi(x_m-\bar{x})\int_0^{2\bar{x}} x^m dx = \varphi(x_m-\bar{x})\frac{(2\bar{x})^{m+1}}{m+1}, \quad m \neq -1,$$

in which $0 < x_m < 2\bar{x}$. Making use of this relation in (5), we find that

$$r = \frac{\varphi(x_{n+1}-\bar{x})\dfrac{(2\bar{x})^{n+2}}{n+2} - \varphi(x_n-\bar{x})\bar{x}\dfrac{(2\bar{x})^{n+1}}{n+1}}{[\alpha(\bar{x})-2\bar{x}\varphi(\bar{x})]^{1/2}\left[\varphi(x_{2n}-\bar{x})\dfrac{(2\bar{x})^{2n+1}}{2n+1} - \dfrac{1}{\alpha(\bar{x})}\varphi^2(x_n-\bar{x})\dfrac{(2\bar{x})^{2n+2}}{(n+1)^2}\right]^{1/2}}, \quad (6)$$

the values x_n, x_{n+1}, x_{2n} being between 0 and $2\bar{x}$. Equation (6) is easily reduced to the form

$$r = \frac{\varphi(x_{n+1}-\bar{x})(2\bar{x})\dfrac{n+1}{n+2} - \varphi(x_n-\bar{x})\bar{x}}{[\varphi(\bar{x})-2\bar{x}\varphi(\bar{x})]^{1/2}\left[\varphi(x_{2n}-\bar{x})\dfrac{1}{2\bar{x}}\dfrac{(n+1)^2}{2n+1} - \dfrac{1}{\alpha(\bar{x})}\varphi^2(x_n-\bar{x})\right]^{1/2}}.$$

Since the function φ is finite and different from zero, it can be seen that r approaches zero as n increases without limit.

In order to consider the case $n \leqq -\frac{1}{2}$, we set $n = -m$ and write

$$r = \frac{\displaystyle\int_\epsilon^{2\bar{x}-\epsilon}(x^{-m+1}-\bar{x}x^{-m})\varphi(x-\bar{x})dx}{\left[\displaystyle\int_\epsilon^{2\bar{x}-\epsilon}(x^2-\bar{x}^2)\varphi(x-\bar{x})dx\right]^{1/2}} \times \left[\int_\epsilon^{2\bar{x}-\epsilon} x^{-2m}\varphi(x-\bar{x})dx - \frac{\left\{\displaystyle\int_\epsilon^{2\bar{x}-\epsilon} x^{-m}\varphi(x-\bar{x})dx\right\}^2}{\displaystyle\int_\epsilon^{2\bar{x}-\epsilon}\varphi(x-\bar{x})dx}\right]^{1/2}.$$

Applying the law of the mean for definite integrals and reducing, we get

$$r\left[\int_{\epsilon}^{2\bar{x}-\epsilon}(x^2-\bar{x}^2)\varphi(x-\bar{x})dx\right]^{1/2}$$

$$\frac{\varphi(x_{-m+1}-\bar{x})}{-m+2}\{(2\bar{x}-\epsilon)^{-m+2}-\epsilon^{-m+2}\}$$

$$-\frac{\varphi(x_{-m}-\bar{x})}{-m+1}\bar{x}\{(2\bar{x}-\epsilon)^{-m+1}-\epsilon^{-m+1}\}$$

$$=\frac{}{\left[\frac{\varphi(x_{-2m}-\bar{x})}{-2m+1}\{(2\bar{x}-\epsilon)^{-2m+1}-\epsilon^{-2m+1}\}\right.}$$

$$\left.-\frac{1}{\alpha(\bar{x}-\epsilon)}\frac{\varphi^2(x_{-m}-\bar{x})}{(-m+1)^2}\{(2\bar{x}-\epsilon)^{-m+1}-\epsilon^{-m+1}\}^2\right]^{1/2}$$

$$\frac{\varphi(x_{-m+1}-\bar{x})}{-m+2}\{\epsilon^{m-(1/2)}(2\bar{x}-\epsilon)^{-m+2}-\epsilon^{3/2}\}$$

$$-\frac{\varphi(x_{-m}-\bar{x})}{-m+1}\bar{x}\{\epsilon^{m-(1/2)}(2\bar{x}-\epsilon)^{-m+1}-\epsilon^{1/2}\}$$

$$=\frac{}{\left[\frac{\varphi(x_{-2m}-\bar{x})}{-2m+1}\{\epsilon^{2m-1}(2\bar{x}-\epsilon)^{-2m+1}-1\}\right.},$$

$$\left.-\frac{1}{\alpha(\bar{x}-\epsilon)}\frac{\varphi^2(x_{-m}-\bar{x})}{(-m+1)^2}\{\epsilon^{m-(1/2)}(2\bar{x}-\epsilon)^{-m+1}-\epsilon^{1/2}\}^2\right]^{1/2}$$

where x_{-m}, x_{-m+1}, x_{-2m} are between ϵ and $2\bar{x}-\epsilon$. It is readily seen that this expression approaches zero with ϵ if $m > \frac{1}{2}$ (*i.e.*, $n < -\frac{1}{2}$), provided m is different from 1 and 2 (*i.e.*, $n \neq -1, -2$). Since $\left[\int_{\epsilon}^{2\bar{x}-\epsilon}(x^2-\bar{x}^2)\varphi(x-\bar{x})dx\right]^{1/2}$, the coefficient of r in the foregoing equation, approaches a limit which is different from zero as ϵ approaches zero, r has the limiting value zero for $n < -\frac{1}{2}$ ($n = -1$ and -2 excepted).

We have now only to consider the special cases $n = -\frac{1}{2}, -1, -2$. For these values of n certain integrals in r become logarithms. It is not difficult to show that for each of these three values r approaches zero with ϵ. The proofs will however be omitted as they can be effected by the methods ordinarily employed in evaluating indeterminate forms.

To summarize: *The correlation coefficient r for the variates x and y ($y = kx^n$, $k > 0$), x being normally distributed, is zero for $n \leqq -\frac{1}{2}$ and for $n = 0$, is negative for $-\frac{1}{2} < n < 0$ and positive for $n > 0$, and approaches the value zero as n approaches ∞; moreover r is equal to unity for $n = 1$ but is less than unity in absolute value for all other values of n.*

COLLEGE GEOMETRY.[1]

By NATHAN ALTSHILLER-COURT, University of Oklahoma.

The ancient Greeks never studied synthetic geometry—at least they never knew they did. Henri Poincaré remarks somewhere in his philosophical writings that of all living creatures inhabiting the earth man alone is mortal—the others do not know that they are to die. The ancient Greeks did not know that the geometry they were studying was synthetic, because they knew of no other kind. The revival of learning in Europe during the Renaissance brought to life the geometry of the Greeks. In the brilliant school of the French mathematicians of the first half of the seventeenth century synthetic geometry had two important exponents, Desargues and de la Hire,—it would be unfair not to mention Pascal also. But the dazzling discovery of analytic geometry made by Descartes and the new avenues opened for it by the invention of calculus entirely absorbed the attention of mathematicians. The contributions of Pascal and Desargues remained unnoticed for a century and a half.

A change occurred at the end of the eighteenth century due to Monge and his pupils: Carnot, Gergonne, Brianchon, Poncelet. Synthetic geometry came again into its own and developed along two distinct lines: projective geometry and modern geometry.

In the last quarter of the nineteenth century modern geometry became enriched by a splendid addition known as the "geometry of the triangle."

Projective geometry is taught widely in the colleges and universities in this country. We can even boast of some very serious contributions to the literature on the subject, as for instance Veblen and Young's two-volume work. But modern geometry has found little or no favor on this side of the Atlantic. For lack of both time and competence I shall not attempt to find a reason for this phenomenon. I shall simply take the liberty to call your attention to the fact, because this state of things seems to me very regrettable.

Here is a body of geometric doctrine, very beautiful in its simplicity, about the existence of which our college students of mathematics have no chance of learning. There are very many things in modern geometry that ought to constitute a part of the mathematical equipment of any college student who takes an interest in mathematics. Take for instance the properties of the radical axis and radical center, Ceva's theorem, the nine-point circle, centers of similitude of two circles, or such more recent things as the Brocard points, the Symmedian lines, and other elementary notions of the geometry of the triangle. With the present day tendency to reduce the high school course in geometry to the very "essentials" (whatever that may mean), the students enter college with a considerably reduced geometric knowledge, as compared with those of former years. They know little of geometric constructions beyond the very elementary ones, and very little or nothing about, say, the escribed circles of a triangle, or the problem of Apollonius.

[1] Read before the Mathematical Association of America at Cincinnati, December 27, 1923.

Aside from its general informational value, the study of modern synthetic geometry in our colleges has yet another very pertinent raison d'être. If we leave out the engineering students, the vast majority of the college students who continue their studies in mathematics beyond their freshman year are prospective teachers of mathematics in our secondary schools. The high school teachers of mathematics of to-day are our students of yesterday, and our students of to-day are the high school teachers of to-morrow. How does their college training prepare them for their task? The traditional course in college mathematics includes algebra, trigonometry, analytic geometry, and the calculus. Aside from the general development of mathematical thinking derived from such a course, the student also gains a great degree of algebraic skill which will help him in his classroom when he teaches algebra. But what specific help does this course give the prospective teacher of high school geometry? A course in projective geometry will widen the geometric outlook of the student. So would a discussion of the foundations of geometry, and far be it from me to belittle the value of these subjects for the prospective teacher. But the methods of projective geometry are totally different from those of Euclidean geometry, and the same is true about the contents of these two subjects. The result is that the prospective high school teacher, upon his graduation from college, knows about Euclidean geometry and its methods of proof exactly as much as he knew when he completed his high school course in this subject. Indeed, he knows much less, because he has had time to forget most of it, since his college studies made but little direct appeal to his knowledge of elementary geometry. When confronted with a problem in plane geometry, he will have only his own resources to fall back upon, since his college professors have done nothing, at least directly, to help him in his task. It is difficult to see how such a state of things can be considered anything short of abnormal. And it appears still more so in view of the fact that there is right at hand a body of doctrine whose study would both extend the field of knowledge of the prospective teacher in the domain of plane geometry and would provide an opportunity for him to review and to give a more mature consideration to his high school geometry; it would teach him to apply the methods of proof and of solving problems which he will use in his classroom with his pupils.

From a course in modern geometry our college students may derive yet other advantages. The traditional college courses, the calculus for instance, are little more than introductions into a vast field, and the student will have to continue to work in such a field considerably more before he can reach the stage where he may find the least opportunity to do something original, something by himself. The vast majority of high school teachers, however, do not go beyond their college course. At the most they may take another year's work. An initiation into modern geometry would suffice to give the student an opportunity to do some little creative work of his own. The results may be modest, but a chance to live through, even on a small scale, the pains of creation and the joy of discovery is much too valuable, much too precious to be given up lightly.

But one may go further. Among all these attempts some real gain for the progress of modern geometry, and thus for the advancement of science, may be

discovered some day. There is no reason why this country should not take the lead in this particular field of mathematical activity. Certainly one may hope that the vast army of high school teachers we have in the United States, a body greater than the corresponding bodies of a good many European countries put together, would be worthily represented in this field of research, if the study of modern geometry should become more widespread among our prospective high school teachers.

Let us now consider the subject from the point of view of the student. However interesting and desirable a change or improvement in the curriculum may be from the point of view of the teacher, the question that we must ask ourselves is: what attitude will the student take toward the innovation? Will the student's appreciation of these things coincide with ours, or will his reactions destroy all the hopes that we had attached to the change? I believe I am in a position to give you some idea of what you may anticipate in this case. For, if I seem to preach a new faith, I am fortunately not in the position of the preacher who does not practise what he preaches.

About six years ago I brought this question to the attention of my colleagues in the University of Oklahoma, and a course in modern synthetic geometry, under the name of "College Geometry," was introduced and has since been conducted under my direction. The course at once found favor in the eyes of the students, and its popularity with those who do their major work in mathematics has been steadily growing. The course is not required, but most students who expect to teach geometry in high school make it a point to enroll in "College Geometry," because they feel the need of it and realize how helpful it will be to them in their high school work.

Still more interesting, perhaps, is the attitude of students in the summer school. In these classes there have been principals of high schools and superintendents of schools, men and women who have been teaching high school mathematics for many years, who would volunteer opinions to the effect that for them as teachers this is one of the most valuable courses they have ever had. Some of them would go so far as to insist that in their opinion no one ought to be permitted to teach high school geometry who has not had a course in "College Geometry." On the intellectual side the subject is nothing short of a revelation to them. They never suspected that right alongside of the geometry they have been teaching all these years there is a direct extension of it, built of the same material, so closely interwoven with the elementary geometry, and yet so interesting, so new, so fascinating. The direct connection between elementary geometry and modern geometry makes most of them feel that there are ample possibilities ahead along these lines, and some of them get the inspiration to try to accomplish something themselves. Last summer one student, a man of middle-age, said to me: "At last I have found something to work upon." I have never had more enthusiastic, more hard-working classes than in "College Geometry."

So far I have spoken of modern geometry as a subject for advanced students in mathematics. I should like to say a few words in favor of it as a freshman sub-

ject. Most of the colleges require the freshmen to take a few hours of mathematics. The courses offered to freshmen are the traditional college algebra and trigonometry. But custom aside, is there any intrinsic reason why a college freshman should continue to study his high school algebra rather than his high school geometry? On narrowly utilitarian grounds neither can be defended, while from a broader cultural and intellectual point of view a continuation of the study of geometry is surely as valuable as an additional course in algebra. One may go still further. In addition to the first year's course in algebra most of the high schools offer the student an opportunity to take another semester's work in the subject. But a year's work in plane geometry seems to be the limit beyond which the inquisitiveness of the student must not be permitted to go. But it would be futile to insist at present on this point. Before we can expect to teach the high school students, we must first prepare the high school teachers, so that we come back to the point where we started.

It is the need of the prospective high school teacher that made me give some thought to the place of modern geometry in the college curriculum. I do not think that the conditions in Oklahoma in this respect are so radically different from those in other states, or in other schools.[1] I was thus prompted to bring the question before the Association in the hope that I may find out what other schools are doing, and I hope that wide discussion will shed upon this subject all the light of which it is worthy.

THE ARITHMETIC CLASSIC OF HSIA-HOU YANG.

By Père LOUIS VANHÉE, S.J., Brussels.

The Chinese mathematician Hsia-hou Yang, a writer of the sixth century, was the author of a brief work, the *Hsia-hou Yang Suan-king*, which has already been mentioned in Professor Smith's *History of Mathematics* (vol. I, p. 150) but which is worthy of a more extended description than such a treatise can be expected to give.

It was one of the standard works demanded for the official examinations in mathematics during the Tang Dynasty (618–907), and has been preserved in the extensive encyclopedia, *Yong-lo*, although the compilers of that work have separated it into parts for the purpose of supplementing the much older treatise, the *K'iu-ch'ang Suan-shu* (*Arithmetic in Nine Sections*). Fortunately the index is so arranged as to allow one to determine the original arrangement of the material, and to this fact we owe the *editio princeps* as prepared in 1776 by the celebrated Peking scholar, Tai Chen. So important was this edition considered that the emperor Khien-Lung wrote the preface and supplied most of the money necessary for its publication.[2]

[1] In fact, the success of this course at the University of Oklahoma is corroborated by testimony from several other institutions where such a course has been given. EDITORS.

[2] A copy of this rare work is preserved in the Bibliothèque Nationale at Paris. See Courant's catalogue, No. 4.844.

The work is divided into three parts, spoken of as the higher, middle, and lower sections, and consists of only thirty-six leaves (double pages), of which the several sections have respectively twelve, thirteen, and eleven folios, every page containing nine vertical lines of about twenty-one characters each. Allowing for blank lines and for spaces, there are only about 12,000 words in the entire work. As in all the older Chinese books, no technical rules are given, and the problems are simply followed by the answers, occasionally with very brief explanations.

In the first section the five operations of addition, subtraction, multiplication, division, and square and cube roots are given. The work on division is subdivided into (1) "ordinary division"; (2) "division by ten, hundred, and so on," especially intended for work in mensuration; (3) "division by simplification" (*yo ch'u*). The last problem in the section is as follows:

"There are 1843 *k'o*, 8 *t'ow*, 3 *ho* of coarse rice. A contract requires that this be exchanged for refined rice at the rate of 1 *k'o*, 4 *t'ow* for 3 *k'o*. How much refined rice must be given?" The answer is 860 *k'o*, 534 *ho*. The solution is given as follows: "Multiply the given number by 1 *k'o*, 4 *t'ow* and divide by 3 *k'o* and you will obtain the result." [1]

Fractions are also mentioned, special names being given to the four most common ones, as follows:

$\frac{1}{2}$ is called *chung p'an* (even part);
$\frac{1}{3}$ is called *shaw p'an* (small part);
$\frac{2}{3}$ is called *thai p'an* (large part);
$\frac{1}{4}$ is called *joh p'an* (weak part).

In the second section there are twenty-eight applied problems relating to taxes, commissions, and such questions as concern the division by army officers of loot and food (silk, rice, wine, soy sauce, vinegar, and the like) among their soldiers.

The third section contains forty-two problems each beginning with the word "now," which is here taken as substantially equivalent to the word "if." Five of these problems, translated as literally as possible, are as follows:

Ex. 1. Now for 1 pound of gold one gets 1200 pieces of silk. How many can you get for 1 ounce? Answer: For 1 ounce you get exactly 75 pieces. Solution: Take the given number of pieces, have it divided by 16 ounces, and you will obtain the answer.[2]

Ex. 2. Now you have 192 ounces of silk. How many *choo* have you? Answer: Four thousand six hundred eight.[3]

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[1] 1 *k'o* = 10 *t'ow* = 100 *ho*. The *k'o* may be roughly translated as a bushel.

[2] The Chinese pound was, from early times, divided into 16 ounces or *taels*.

[3] The result (4608) is written in words instead of in numeral characters. The zero was not n use in the sixth century; but this did not, of course, prevent the use of the ordinary Chinese numerals of that period. The answer shows that the ounce was at that time divided into 24 *choo*.

At the beginning of most of the early Chinese books on arithmetic there is the following statement from Sun-tzĭ's *Suan-king*, a work referred to in Smith's *History of Mathematics*, vol. I, p. 141: "Weight begins with one grain of millet; ten grains make one *ts'en;* ten *ts'en* make one *choo;* 24

Ex. 15. Now 2000 packages of cash must be carried to the town at the rate of 10 cash per bundle. How much will be given to the mandarin and how much to the carrier? Answer: 1980 packages and $198\frac{2}{101}$ cash to the mandarin; 19 packages and $801\frac{98}{101}$ to the carrier. Solution: Take the total number as the dividend, and 1 package plus 10 cash as the divisor.[1]

Ex. 24. Out of 3485 ounces of silk how many pieces of satin can be made, 5 ounces being required for each piece? Answer: 697. Solution: Multiply the number of ounces by 2 and go back by one row. Dividing by 5 will also give the answer.[2]

Ex. 42. Now they build a wall, high 3 rods, broad 5 feet at the upper part and 15 feet at the lower part; the length 100 rods. For a 2-foot square a man works 1 day. How many days are required? Answer: 75,000. Solution: Take half the sum of the upper and lower breadths, have it multiplied by the height and length; the product will be the dividend. As the divisor you will use the square of the given 2 feet.[3]

Hsia-hou Yang also uses percentage. He shows considerable ability in finding various areas and volumes. His work is evidently a good type of practical textbook of the time, a fact that is shown by its popularity and the high esteem in which it has always been held. Yüan Yüan,[4] for example, in his well-known biographical work, speaks of it as "an easy text, intended for daily use." We thus have a fair idea of the elementary calculations performed by the Chinese in the sixth century, and a satisfactory basis for comparison of the oriental ability with that of the contemporary and barren period in the West.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. A NOTE ON KNOTS.

By F. V. MORLEY, New College, Oxford, England.

1. The construction of a regular pentagon by tying a simple knot in a strip of paper leads directly to a generalization for the construction of regular polygons of any odd number of sides.

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choo make one ounce (*tael*); 16 ounces make one pound; 30 pounds make one *kiun*; 4 *kiun* make 1 stone.

[1] A package or string of cash contains 1000 farthings. The two results reduce to periodic fractions. There is no reason given for the division of 2,000,000 by 1010.

[2] This is an early use of our rule for dividing by 5. The expression about going back one row seems to refer to the use of counters. So far as known, the *suan-pan*, in its present form, was not yet invented.

[3] 1 rod = 10 feet = 100 inches. The amount made by one man in a day is evidently intended to be the volume of a rectangular solid of base 2 ft. square and of height 1 rod; that is, 40 cubic feet. The divisor, therefore, is 40.

[4] See Smith, *loc. cit.*, vol. I, p. 535. Yüan Yüan was born in 1764 and died in 1849.

The construction of the pentagon came to me by oral tradition, and I am at a loss for a reference to it.[1] Though no doubt familiar to many, it may be recapitulated here. To perform the operation, take a strip of smooth, pliable paper, sides cut parallel and even, say half an inch wide and ten inches long. Visualization may be aided by Fig. 1. The process is simpler in action than in words.

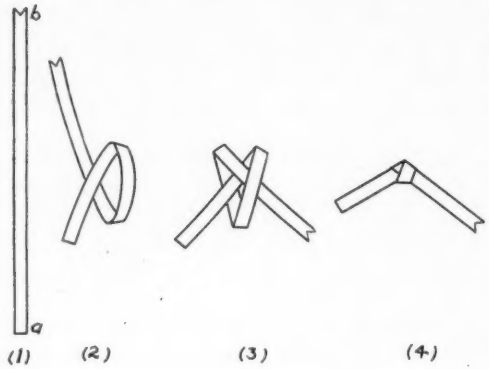

FIG. 1.

1. Hold the strip vertical, the lower end, *a*, between thumb and forefinger of the left hand; the upper end, *b*, in the right hand.

2. Carry the end *b* forward and pass it, from the right, behind *a*—catching the double thickness between the left thumb and forefinger. We have thus a simple loop, with long end *b* projecting to the left.

3. Carry *b* forward again, and pass it from the left axially through the loop.

4. Now pull ends *a* and *b* delicately. Folding the paper neatly flat when the knot is tight, we have the pentagon.

If, with a somewhat thinner strip of the same length, say a quarter of an inch wide, we start from stage 3 in the above process, and, instead of pulling the ends, pass *b* again back and behind the double thickness—so as now to catch three thicknesses between the left thumb and forefinger—we have a double loop, with long end *b* projecting to the left. This is shown in Fig. 2 (5). Carry *b* forward and pass it from the left axially through the double loop (Fig. 2 (6)). Pulling both ends of the strip until the knot is tight, and flattening, we have the regular heptagon (Fig. 2 (7)).

FIG. 2.

This construction applies to any regular polygon of $2n + 3$ sides, where n is the number of loops in the knot. Thus from Fig. 2 (6) we might, instead of pulling the ends, form a triple loop by passing *b* back and behind. Tying the knot with this triple loop, the result will be a regular nonagon; and so on.

2. The question rises of constructing the even regular polygons by knots. I do not think this can be done with a single strip of paper. Whenever we tie a "four-in-hand" tie, we construct a hexagonal knot; but it does not flatten into a regular hexagon.

[1] Mr. H. W. Richmond tells me that the construction is mentioned in *Scientific Amusements*, by Tom Tit; translated from the French by C. G. Knott, and published by Nelson (no date mentioned).

However, we may take two strips, of equal width, and make a simple loop in each, as in Fig. 1 (2). Call the ends of the strips a, b and α, β respectively. Turn the second loop over, and join it with the first as shown in Fig. 3 (8); that is, with ends α, β through the loop of a, b, and ends a, b through the loop of α, β. Now pulling the ends, and flattening, we get the regular hexagon (Fig. 3 (9)).

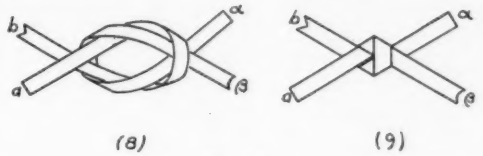

FIG. 3.

If, instead of using two simple loops, we had in this way combined a double loop and a single loop, the result would have been the regular octagon. Combining a triple loop and a single, we get the regular decagon. And so on.

3. The above constructions give any regular polygon of five or more sides. Physically, the strips are not easy to manipulate when the loopage is high. Theoretically the construction may be thought of as that of tying knots in parallel lines, and may so continue *ad infinitum*. If an analysis can be developed to handle such processes of knotting, we shall have, in a treatment of these constructions, a method for solving particular equations of any degree.

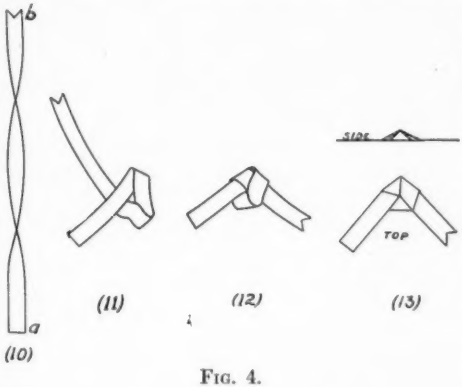

FIG. 4.

There are plenty of possible complications to be studied. One interesting development is to twist the strip. Here it is well to have one face colored, to aid in seeing what happens. If a is held and b twisted through two right angles perpendicularly to the length of the strip—so that b presents the same face as a—we say the strip has one complete twist (Fig. 4 (10)). Tying the pentagonal knot in a strip once twisted, we have in Fig. 4 (12) a knot which will not flatten into a plane polygon; but it "flattens" naturally on a pentagonal pyramid (Fig. 4 (13)). In other words, a simple knot tied in a helical strip may give a pentagonal pyramid. In the strict sense I have no proof, other than having seen it happen.

II. THE DEFINITION OF "VARIABLE."

By A. A. BENNETT, University of Texas.

Among the important tools of mathematicians is to be reckoned the concept of a variable. A search through mathematical treatises brings out the fact that a definition of this term is more often omitted than included. In works entitled

"Theory of Functions of a Real Variable" or "Theory of Functions of a Complex Variable," the term "function" is carefully defined with numerous cautions and illustrations while the idea of a variable is apparently presupposed.[1] This is consistent when the treatise in question also presupposes a familiarity with algebra, analytic geometry and calculus. However not a few laborious tomes start with the first elements of theoretical arithmetic and continue with a supposedly complete set of axioms and definitions, but never define or explain the notion of a variable.

Strictly speaking, algebra as the study of equations and of rules of operation does not require the use of a variable, useful as this notion can be in graphical suggestiveness; but in calculus at least, the idea of variable would appear essential. Yet relatively few books on the calculus and fewer still on analytical geometry suggest a definition of the term. Some careful writers of mathematical treatises explain the term at some length without giving any concise definition. This may be due to a willingness to describe the meaning of the word in an implicit fashion, its properties being implied from significant features of its use in various connections. Such discussions, however grateful, do not meet the logical demand for a definition. One reason justifying this general neglect is that there is little occasion for confusion in the use of the word "variable." The concept is suggested with reasonable vividness by the etymology, and paradoxes based on its use would be hard to find. It might therefore be with some surprise that one notes considerable discrepancies among available definitions, and that most of the familiar formulations are open to objection.

A familiar definition is the following: "A variable is a symbol that denotes more than one quantity in the course of a single discussion." There are two features of this definition that we might criticize. (1) Scholarly writers at least fail to agree that a variable is but a *symbol* of a quantity. To be sure, Keyser in his *Mathematical philosophy* insists at some length that such is the case. Pasch in *Veränderliche und Funktion* refers to "a variable as a number-name. The individual numbers which the number-name may denote are the values of the number-name. It is permissible to understand by the symbol of a variable each arbitrary individual among its values." Hardy in *Pure mathematics* calls a variable "an unspecified element" in a given "field of variation," although he makes no explicit definition. Hobson in *Theory of functions of a real variable* refers to the "essential nature of the variable (as) consisting in its being identifiable with any particular number of the domain," although again there is no concise definition of the word under discussion. Quite irrespective of authority, it is clear that the language used in connection with variables does not suggest any underlying concept of the variable being a symbol other than a quantity of the same sort as the constants entering the problem. The sum of two variables is never regarded as requiring special explanation as would be the case were the variables new symbols, for which in the nature of things there could be no in-

[1] One may note that Caratheodory's *Vorlesungen über Reelle Funktionen* contrives to avoid the word "variable" altogether until page 641, when it slips in undefined.

herent rules of addition. One does not speak of the limit of a variable as the limit of a symbol, and so on through the list of customary locutions. With some care the treatment of a variable as a symbol might be made consistent but it would appear artificial and in fact seems never to have been undertaken. One might well insist that a variable is best left undefined, merely the term "symbol of a variable" being defined, and the use of the term "variable" by itself being justified as a simple and obvious abbreviation in the several phrases in which it appears. (2) One should insist upon the disjunctive application of the term, while the expression "more than one" might well refer to a set taken as a whole, that is, conjunctively. In plane analytical geometry, a point may be represented by a pair of numbers, and therefore by the single symbol (a, b), but not many persons would desire to regard this symbol as a variable simply because it represents more than one number, namely a and b. On the other hand a typical variable, $y = f(x)$, may for some choice of the function f denote but a single number, the constant value of the function. It would seem that any definition of "variable" to be acceptable should cover the special case of a constant. However this seems to be a point about which there is not unanimity. For instance, Fine in his *College algebra* specifically excludes the constant case and refers to a variable as a "letter . . . free to take every possible value and to change from any one value to any other."

A revised definition of a variable as "any one of a set of elements" avoids indeed the conjunctive ambiguity and escapes likewise the awkwardness of regarding a variable as merely a symbol. Much confusion might still result from the ambiguity of the phrase "any one." In one use there is the notion of uncertainty, in another that of free choice. When a variable is used to refer to "any one," it is essential that the reader does not obtain the impression, that might be justified by the words, that a variable is "some one not as yet revealed to the reader but perhaps already chosen by the author, and which the author reserves the privilege of identifying at some later stage of the discussion"; but rather that it is something better expressed as follows, "an unspecified one of a set (the set being mutually agreed upon by author and reader) which the reader is always at liberty to choose in whatever manner he may desire from the given set, but which, for illustrative purposes only, the author may request to be identified in turn with one or more specific elements." The fact that there is no time-factor involved in the notion might seem self-evident. However, Fricke in *Hauptsätze der Differential- und Integral-Rechnung* speaks as follows: "A quantity which in the course of time assumes different values is called a variable quantity, or briefly a variable."

I propose the following definition which presents the notion in an implicit form. Explicit definitions when convenient are of course to be preferred on grounds of elegance. There are however many terms which it seems best to leave not completely defined by explicit statement. Some writers in their effort to make all definitions explicit do violence to current mathematical language. It would be possible to define "point," if algebraic analysis be presupposed, a point being

identified with its set of coördinates. It is possible to define "vector" as a set of point pairs, but since only equality of vectors and the result of combining vectors is essential, it would seem better to leave the term defined only implicitly. Certainly the remark that a given point pair is an element of a vector sounds bizarre. In the same way it does not seem necessary to know just what an irrational number is, whether it is an infinite series, a cut in the number system, an upper bound of a set of rational numbers, etc., so long as we call it a number, and know its properties in connection with other rational and irrational real numbers. The explicit question as to whether a variable is or is not a number may have philosophical interest but does not need to be decided for the investigation of mathematical results. It is in the same position as the question as to whether a rational fraction is a pair of integers, concerning which we need merely insist that a rational fraction is a number and is in one-to-one reciprocal relation to pairs of integers subjected to certain familiar rules and conventions. One might mention incidentally that the phrase "in turn," which is sometimes used in connection with the definition of variable, is obviously inapplicable to the usual case for which the total set is non-enumerable.

Definition: A symbol that denotes any one you please of an assigned set of elements may be called the symbol of a variable element of this set, or more briefly of a variable, when there is no ambiguity as to the set.

It is customary in mathematics to use language that would suggest that the symbol is identified with the object symbolized. The statement "5 is a number" is not to be criticized on the ground that "5" can be nothing other than a mark or token while a number is an abstract concept. A more detailed statement would be "The symbol 5 is the symbol of a number." But such usage is authorized by long and general practice. The language "x is a variable" would be in the same manner consistent with our definition, although a more elaborate statement of the same content would be "The symbol x is the symbol of a variable." We will be content to regard the term "variable" as referring to an abstract, and one might insist undefined notion, so long as one will agree to use it only in connection with the words "symbol of" either expressed or, as is usually the convention, only implied by the context, the entire phrase "symbol of a variable" being defined as above.

RECENT PUBLICATIONS.

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REVIEWS.

Vector Analysis. By C. RUNGE, translated by H. LEVY. New York, E. P. Dutton and Co., 1923. 8vo. viii + 226 pages.

This volume by Professor Runge of the University of Göttingen is a logical development of the subject of vector analysis and forms an excellent introduction

to the application of vectors in many fields. Several pages are devoted to the relation between determinants and vector theory and sufficient of the theory of lattices is given to form a basis for a study of the lattice structure of crystals. Since the author is a mathematical physicist, it is quite natural that he should confine his attention mainly to the realm of physics, the sphere to which vectors seem essentially to belong. Hydromechanics, electromagnetic theory, and elasticity are subjects of frequent reference and in the discussion of tensors there is an admirable introduction to the mathematical theory of relativity.

Chapter I introduces the vector idea and the rules of vector algebra. The author employs a slight variation from the usual method of approach. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors, then $\boldsymbol{ab}$ is called the external product of $\boldsymbol{a}$ and $\boldsymbol{b}$ and is termed a vectorial area of definite sense and of magnitude equal to that of the parallelogram of which $\boldsymbol{a}$ and $\boldsymbol{b}$ are sides, drawn in, or parallel to, the plane determined by $\boldsymbol{a}$ and $\boldsymbol{a} + \boldsymbol{b}$. A vector $\boldsymbol{f}$ perpendicular to $\boldsymbol{ab} \equiv \boldsymbol{F}$ is the representation or complement of $\boldsymbol{F}$, where $\boldsymbol{f}$ is of such magnitude that its absolute value is equivalent to that of $\boldsymbol{F}$. This relationship is written $\boldsymbol{f} = |\boldsymbol{F}$ or $\boldsymbol{F} = |\boldsymbol{f}$. With this beginning, the scalar and vector products of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ may be defined $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}|\boldsymbol{b}$ or $\boldsymbol{b}|\boldsymbol{a}$ and $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{ab} = -|\boldsymbol{ba}$. The external product $\boldsymbol{abc}$ represents in a defined sense the volume of a parallelopiped of edges $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. The external product of two vectorial areas is defined and several pages are devoted to reciprocal vectors and their uses.

In chapter II, the rules for differentiation and integration of vectors are defined and the rules for the use of the operator ∇. Applications are made to space curves, surfaces and volumes. Theorems are developed for the transformation of surface into volume integrals and of line into surface integrals. Green's theorem, rotors and potential as applied to scalars, vectors and vector areas are discussed.

In chapter III, the affine transformation of space is defined as that transformation by which a vector $\boldsymbol{r} = x\boldsymbol{a} + y\boldsymbol{b} + z\boldsymbol{c}$ from a fixed point O to a point R transforms into the vector $\boldsymbol{r}' = x\boldsymbol{e} + y\boldsymbol{f} + z\boldsymbol{g}$ from O to the new position R' of R, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{e}$, $\boldsymbol{f}$, $\boldsymbol{g}$ being arbitrary vectors, but the scalars x, y, z being the same in both cases. If $\boldsymbol{a}'$, $\boldsymbol{b}'$, $\boldsymbol{c}'$ are the reciprocal vectors of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ respectively, then the operator $\boldsymbol{T} = \boldsymbol{ea}' + \boldsymbol{fb}' + \boldsymbol{gc}'$, which effects this transformation, is termed a tensor. Different types of tensors are then discussed and rules for their use set forth. Cogredience and contragredience end the first volume. The author states in the introduction "This first volume contains the vectorial analysis of three demensions. In the second volume that of four and more dimensions playing an important part in the theory of relativity will be treated."

The reader may question whether all of the notation is necessary but it is sufficiently near to that already recognized so that little difficulty will be experienced in getting accustomed to it. Misprints are few. On page 61, line 17, the second = sign should read $-$ and dr, line 12, page 92, should be $d\tau$. The translation is well done. One encounters a few ambiguous and careless statements, such as the last sentence of paragraph 14, page 38: "The vectorial

product of a vector similarly or oppositely directed is zero since their external product is zero."

This book is a valuable addition to existing texts in English on vector analysis, being comprehensive and meaty yet not too formidable. It is full of suggestions of interesting lines of investigation and should attract the attention of all who desire a modern and authoritative presentation of the subject.

J. B. REYNOLDS.

A History of Magic and Experimental Science during the First Thirteen Centuries of Our Era. By LYNN THORNDIKE. New York, The Macmillan Co., 1923. Vol. I, pp. xl + 835; Vol. II, pp. vi + 1036.

This history, in two massive volumes, is truly a magnum opus. It represents many years of painstaking research among medieval manuscripts in European libraries, as well as the careful weighing of the judgments of older writers on medieval magic and science. Much of the source material examined has not previously been used. This book affords detailed information on writers not readily accessible to most students of history. For example, the reader finds here sixty pages devoted to Pliny's natural philosophy, sixty-five pages to Galen, nineteen to Augustine, thirty-three to "The Pseudo-Aristotle," seventy-six to Albertus Magnus, and another seventy-six pages to Roger Bacon. In other words, the treatment of the leading characters is the equivalent of a small book on each. Another way of describing the contents of the work is to state that the attitude toward magic and science is described for the period of the Roman empire and early Christianity, for the early Middle Ages, for the twelfth and the thirteenth centuries.

Primarily the book is of interest to students of natural philosophy and astronomy, rather than to mathematicians. Yet even the latter will discover points of interest. Since the time of Benjamin Peirce the precise definition of mathematics has repeatedly interested the votaries of that science. Hence the early conceptions of the words mathematics and mathematicians are of interest. In the Roman empire, astrologers were called mathematicians, as is disclosed in the "Recognitions," a book ascribed to Clement of Rome. The western church father Tertullian at the beginning of the third century rejoices that the mathematici or astrologers are forbidden to enter Rome or Italy, the reason being that they are consulted so much in regard to the life of the emperor. Augustine, in the fifth century, refers to the mathematici as astrologers and holds that they enslave human free will by predicting a man's character and life from the stars. Similarly, Isidore of Seville refers to the mathematici who augur the future from the stars, assign the parts of the soul and body to the signs of the zodiac, and try to predict the nativities and characters of men from the course of the stars. An anonymous author of the tenth century refers to the wisdom of the mathematici who think that mundane affairs are carried on under the rule of the constellations. This conception of mathematici was held by Marbod, the Bishop of Rennes, as late as the beginning of the twelfth century. Saint Hildegard of

Bingen called mathematici "deadly instructors and followers of the Gentiles in unbelief."

Gradually a distinction came to be drawn between *mathematica* or mathesis and *matematica* or matesis. The historian Richer, a contemporary of Gerbert, apparently used the word mathesis in the sense of our modern mathematics. Over a century later, Vincent of Beauvais, a contemporary of Abelard, lets mathematica mean sound doctrine and (as with Aristotle) the science of abstract quantity, while matesis signifies that superstitious vanity which places the fall of man under the constellations. Similar distinctions were made later by William of Conches, John of Salisbury, Albertus Magnus and Roger Bacon. Michael Scott of the court of Frederick II draws a very sharp distinction between mathesis or knowledge, and matesis or divination, also between mathematica which may be taught freely and publicly, and matematica which is forbidden to Christians.

Modern mathematicians will be amused by the web of mysticism which ancient thinkers wove around particular numbers. There was a fusion of Pythagorean and Hebrew number mysticism in the writings of Philo Judæus who emphasized the glories of the number 7. He notes that there are 7 planets, 7 circles of heaven, four quarters of the moon of 7 days each, 7 stars in the Pleiades and 7 in the Ursa major, that children born at the end of 7 months live, that the 7th day in disease is critical, that there are 7 ages of man's life, that the lyre has 7 strings, speech has 7 vowels, also that there are 7 divisions of the head (eyes, ears, nostrils, mouth), etc. However 4 and 6 yield little to 7 as mystic symbols. Hippolytus associates the beneficent and masculine with odd numbers, the feminine and malicious with even numbers. Macrobius held the Pythagorean doctrine that the world-soul consists of number, that number rules the harmony of celestial bodies. Puerile reveries occur in Robert Grosseteste: Form is represented by 1, matter by 2, composition by 3, the compound by 4; now $1 + 2 + 3 + 4 = 10$. Wherefore, every perfect thing is 10.

Thorndike utters the philosophical tenet that "magic and experimental science have been connected in their development, that magicians were perhaps the first to experiment." He gives the term "magic" a broad interpretation, "including all occult arts and sciences, superstitions, and folk-lore." He states further that "magic implies a mental state and so may be viewed from the standpoint of the history of thought." Thorndike does not press his philosophic views. It seems to the reviewer that the existence of a causal relation between magic and science is not evident. Did magic appreciably stimulate experimental science? Did number mysticism lead to any real theorems about numbers? Did astrology contribute substantially to the progress of astronomy? Did the horse of Bellerophon which according to legend was suspended in the air by magnets contribute to an exact knowledge of magnetic attraction and mechanical equilibrium? What useful medical practice grew out of the belief of Pliny that a man who has been struck by lightning will speak at once, when turned over on his injured side? What stimulus to science grew out of the claim that sometimes when a crocodile opens its jaw an ichneumon "darts down its throat like a javelin and eats away its intestines"?

Indeed, if we may judge the past by the present, mysticism does not lead to science. On the contrary, such new results of science as appeal to popular imagination are prone to stimulate the undisciplined reason of mystics to fresh absurdities. The mathematician's fourth dimension of space has afforded spiritualists rich fields of speculation. Ethereal waves of light have suggested to telepathists easy explanations of alleged thought-transference. The electron theory assists in certain mystic medical practice. Daily weather-bureau forecasts tempt ambitious prophets to predict the weather weeks and months in advance.

Whether Thorndike's philosophical tenet be valid or not, the rich historical material collected in his book and presented to the reader in attractive diction is of immense value. The publication is an epoch-making contribution to the better understanding of the scientific and pseudo-scientific thought of the first thirteen centuries of our era.

FLORIAN CAJORI.

The Rhind Mathematical Papyrus, British Museum, 10057 and 10058, Introduction, Transcription and Commentary by T. ERIC PEET. The University Press of Liverpool Limited, Hodder & Stoughton, London, 1923, 2 + 136 folio pages + 24 plates (3 of which are folding). Price 63 shillings.

Almost all of our knowledge of Egyptian mathematics is derived from: (*a*) four papyri in hieratic writing, and two wooden tablets the contents of all of which date from the twelfth dynasty [1] which began about 2000 B.C.; (*b*) a Demotic papyrus, Byzantine and Coptic tables of fractions, and the Akhmîm papyrus of much later date. On account of the possibility of contamination from Greek mathematics, these may not be used to prove anything with regard to mathematics of the earlier Egyptian periods.

The best known of the papyri is the one found at Thebes in the ruins of a small building near the Ramesseum. It was purchased in 1858 by A. Henry Rhind and after his death it passed into the hands of a gentleman from whom it was purchased by the trustees of the British Museum in 1864. This Rhind mathematical papyrus is in two parts of the same width, 33 cm., and of length 206 cm. and 319 cm. respectively. These two parts were originally one, about 543 cm. in length. Remarkable to relate, a number of fragments belonging to the 18 cm. missing in the British Museum papyrus became in 1907 the property of the New York Historical Society.[2] They were part of the collection of Edwin Smith, were in his possession in 1862–63, and were found in Egypt with a medical papyrus now well known.

[1] Of other material for the study of mathematics, especially weights and measures, various account papyri are of importance. The most valuable of those already published are Papyrus Bulaq 18, and those discussed by Spiegelberg in his *Rechnungen aus der Zeit Setis I*.

[2] The discovery that the New York fragments were originally part of the British Museum papyrus was only made within the past twenty-two months. When examining the Society's papyri in the autumn of 1922, the British Egyptologist, Percy E. Newberry, took tracings of the fragments, since he suspected connection with the Rhind Papyrus. He brought them to the attention of his colleague of the University of Liverpool, Professor Peet. These facts were kindly furnished to me by Mrs. Caroline R. Williams of the New York Historical Society.

According to Professor Peet, the Rhind papyrus was written between 1756 and 1612 B.C. but it is copied from an older work in the time of a king of the twelfth dynasty who was on the throne from about 1849 to 1801 B.C.

A second papyrus of this dynasty is the one recently acquired by the Museum of Fine Arts in Moscow. It was first described [1] in print in 1917 as containing 19 "problems, some of which give us new types of calculation unknown till now and therefore somewhat difficult to comprehend. Four of these problems are geometrical ones. The first shows how to define the length of the sides of a quadrilateral, when the relation of the sides and area of the quadrilateral is known. The two next give a method of calculating the area of a triangle: a method already known to us." The fourth geometrical problem appears to indicate a familiarity with the formula for the volume of a frustum of a square pyramid, $V = (h/3)(a_1^2 + a_2^2 + a_1a_2)$, where h is the altitude of a frustum, the sides of whose bases are [2] a_1 and a_2. If it is verified that such a result was actually known to Egyptians of the twelfth dynasty, it will lead to an entire revision of our ideas as to the geometrical powers of the Egyptians. Writing in August, 1922, the director of the Moscow Museum informed me that at a congress of Egyptologists, held at Moscow in that month, it was decided that his mathematical papyrus was to be published by the Russian Academy of Sciences, with a translation by Professor W. Strouwé, of Petrograd, and a mathematical commentary by Professor Zienserling.

The third papyrus of the twelfth dynasty consists of fragments found at Kahun in 1889 and published in 1898. Among its contents are: a table of resolutions of all fractions, 2/3 to 2/21, with numerator 2, and denominator an odd number into the sum of two or more fractions with numerator unity; a problem dealing with parallelopipedal containers the sides of whose bases are to one another in a fixed ratio (the solution involves the use of square root); and accounts of a poultry yard.

The fourth papyrus of the twelfth dynasty is Berlin Papyrus 6619, described by Schack-Schackenburg in 1900. One of its problems is to divide 100 square

[1] B. A. Touraeff, *Ancient Egypt*, 1917, pp. 100–102. The attention of mathematicians was first drawn to this by the writer at the Rochester meeting of the Association in 1922.

[2] Brahmagupta (who flourished about 628 A.D.) gave a formula which reduces to this (*Algebra, with Arithmetic and Mensuration, from the Sanscrĭt of Brahmagupta and Bhâscara*, ed. by Colebrooke, London, 1817, pp. 312–313). So also Mahāvīrācārya (c. 850 A.D.), *The Gaṇita-Sāra-Sangraha* . . . with English translation and notes by M. Raṅgācārya, Madras, 1912, p. 260.

In the Moscow papyrus $a_1 = 4$, $a_2 = 2$, and $h = 6$. In al-Khowârizmî's algebra, the same problem is solved for $a_1 = 4$, $a_2 = 2$, and $h = 10$. (*The Algebra of Mohammed Ben Musa*. Edited and translated by F. Rosen, London, 1831, pp. 83–84); the method of solution is: determine the height (20), and hence the volume, of the completed pyramid, from which is subtracted the volume of the pyramid of height 10 on the upper base. To Democritus (who flourished about 400 B.C.) is due the discovery that the volume of a pyramid is one third that of the prism having the same base and equal height; the first rigorous proof of this result was given by Eudoxus.

The general formula for the volume of a frustum of a pyramid $(h/3)(A_1 + A_2 + \sqrt{A_1A_2})$, A_1 and A_2 being the areas of the bases, seems to have been first given by Leonardo Pisano in his 'Practica Geometriæ' of 1220 (*Scritti di Leonardo Pisano* . . . , vol. 2, Rome, 1862, pp. 177–178).

Heron of Alexandria (Third century A.D.?) discusses the volume of a frustum of a pyramid for which $a_1 = 24$, $a_2 = 16$, $h = 50$. *Heronis Alexandrini Opera quae supersunt omnia*, Leipzig, vol. 3, 1903, pp. 115–116).

cubits into two squares whose sides are in the proportion 1 : 3/4; it involves the use of square root. Another problem in numbers involves the correct determination of the square root of $6\frac{1}{4}$ as $2\frac{1}{2}$.

If such mathematics was known in the twelfth dynasty, it is interesting to speculate how much further we must go back in order to find the beginning of its development. Only a very few facts are known. The decimal system of notation for numbers up to 1,000,000 goes back to the first dynasty (which Breasted in 1906 set as beginning about 3400 B.C.), and there are faint traces of the system's having been originally quinary. In the fourth dynasty (which Breasted in 1906 set as beginning about 2900 B.C.) "land measures of the Rhind Papyrus are," according to Peet, "already in full development in a form which involves correct determination of the area of the rectangle but not of necessity of the triangle or circle. There appears to be no early evidence with regard to the measures of capacity, though one may almost take it for granted that with the measurement of the field on which the corn was grown went that of the containers in which it was stored and sold. That measurement by weighing was practised can hardly be denied."

The first published account of the Rhind mathematical papyrus, which is by far the most elaborate of the hieratic mathematical papyri which we have, was by F. Lenormant [1] in 1867. Then ten years later appeared the valuable German translation and commentary of August Eisenlohr who was assisted in the mathematical work by his brother Friedrich and by Moritz Cantor. In 1872 the British Museum loaned to Eisenlohr a set of plates of the Rhind papyrus. He published tracings of these in a volume accompanying the above-mentioned work. But the splendid facsimile issued by the Museum in 1898 is incomparably superior to Eisenlohr's volume of plates.

During the past 50 years the bibliography of the Rhind papyrus and of Egyptian mathematics has become quite extensive; in 1922 the writer prepared an annotated list of about 100 titles, in nine languages. Whilst some of these titles refer to descriptions, in articles or books, of a trivial nature, there are many others indicating notable advance in the understanding of the papyrus due to vastly extended knowledge of the Egyptian language. Hence for some time Eisenlohr's work has been in need of fundamental revision and modification.

It will be a source of profound satisfaction to many that the new work on the Rhind mathematical papyrus is in the English language. The author is a distinguished scholar, the Brunner professor of Egyptology in the University of Liverpool, director of the Institute of Archæology in the University, and editor of the *Journal of Egyptian Archæology*.

Professor Peet writes in his preface, "Every attempt has been made to render the book intelligible to the mathematician who has no knowledge whatsoever of the Egyptian language. On the other hand, the Egyptologist with little knowledge of mathematics may enter on it without fear. Egyptian mathematics

[1] *Comptes Rendus . . . de l'Académie des Sciences*, Paris, vol. 65, p. 903.

was a simple affair, and the author has tried throughout to deal with it in its own simple terms without clothing it in a modern dress which is totally foreign to it."

Not only is the book intelligible to the mathematician, but from the beginning to the end it is of extraordinary interest for such a reader. The attractive, lucid, and stimulating style of the commentary and the remarkably thorough, judicial and scholarly character of the work as a whole stamp it as a contribution of very high order to our knowledge of Egyptian mathematics.

Of the large folio pages in Professor Peet's work, 1–32 are occupied by introductory matter; 33–131, by a translation of the text and commentary; 131–132, by a general index; 134–135, by an index of Egyptian words discussed; 136, by an explanation of the plates. The plates *A*–D, F–Y (omitting I) contain a hieroglyphic translation,[1] with notes, of the original hieratic writing. Hieratic and hieroglyphic writing are about equally old, but most Egyptologists read the latter more readily. It is also of considerable interest to the mathematician who is not an Egyptologist; the signs for numbers may be learned in a few moments; a pair of legs walking toward the right is the sign for addition,[2] toward the left, subtraction; a bird with its beak at the ground is translated "to find." Plate E contains a slightly reduced copy of the ends of the parts of the Rhind papyrus at the break,[3] with 24 of the New York fragments placed in their proper order with reference to them. At one side of the plate are reproductions of 23 small unplaced fragments.

The sub-headings of the introductory matter are as follows: Previous work on the papyrus, Description of the papyrus, Date of the papyrus, Contents of the papyrus (pages 4–5), Documents available for the study of Egyptian mathematics (6–9), Date of the origin of Egyptian mathematics (9–10), General character of Egyptian mathematics[4] (10–21), Method of setting out the sums (21–24), Egyptian weights and measures (24–26), Comparison of Egyptian mathematics with Babylonian (27–31), The Greeks in Egyptian mathematics (31–32).

In this last section the most interesting thing is with reference to "harpedonaptai" or "rope-stretchers" of Egypt who according to some mathematical historians (for example, Fink and Ball) were acquainted with the fact that a triangle whose sides were in the ratios of the numbers 3, 4, and 5 contained a right angle and that they constructed right angles accordingly, as ancient Indian, and possibly Chinese, geometers did. Concerning this, Peet remarks:

[1] This translation was not made from the British Museum facsimile but from the original papyrus and is therefore, according to Peet, "at variance with the *Facsimile*. The more important of these cases," he continues, "are noted in the critical notes. To note all minor instances would have rendered the notes far too bulky." Peet does not seem anywhere to make clear how a *facsimile* of a papyrus *can* differ from the papyrus itself. Of course one can guess what is meant.

[2] In the Moscow papyrus this sign means to square. Curiously enough Peet fails to comment in Problem 28 on the fact that this same sign enters to mean *subtraction*.

[3] In making this copy five changes had to be made in rectification of the papyrus itself which had been carelessly patched at the break.

[4] Professor Peet does not specifically note that the Egyptian method of finding the product of two numbers is very often similar to that of the so-called Russian peasant system of multiplication. Compare my note in this MONTHLY (*1918*, 139–142).

"Nothing in Egyptian mathematics suggests that the Egyptians were acquainted even with special cases of Pythagoras' theorem concerning the squares on the sides of a right-angled triangle. That the *harpedonaptai* were land-measurers, on the other hand is most probable, indeed we can even see such persons at work in the pictures on the walls of the Egyptian tombs . . . the very unit by which fields were measured was a 'reel of rope' of 100 cubits in length. . . . This process of land-measuring with a rope, the Egyptian name for which is not known, has been confused by historians of mathematics with the ceremony called . . . 'the stretching of the cord.' This was one of the initial ceremonies at the foundation of a temple. The king, or whoever represented him, took a sighting of the pole-star through a cleft stick, another standing north of him with a plumb-bob attached to a wooden arm. Each then drove a stake in the ground in front of him, and a cord stretched between the two gave a true north and south line and enabled the four corners of the temple to be fixed. Here the stretching of the cord is used not necessarily in measurement, but in the fixing of the orientation. Possibly it was something of this kind which Democritus had in mind when he spoke of the skill of the *harpedonaptai*."

The Rhind mathematical papyrus starts out with a table of resolution of fractions with numerator 2, and denominators the odd numbers 5 to 101, into the sum of two to four other fractions with unity in the numerators. It was one of the New York fragments which extended this table so as to include 2/101 which is given equal to $1/101 + 1/202 + 1/303 + 1/606$. This is the only case in the table that the number in the denominator of a fraction appears again in its resolution. Results of this table are used many times in the problems which follow.

The problems are unnumbered in the original but for convenience were numbered by Eisenlohr. This numbering has been retained by Peet for the same reason, and because they are now current in the literature of the subject.

Numbers 1–40 are problems in arithmetic. Number 3 is: "To divide 7 loaves among ten men": the answer is found to be that each man received $2/3 + 1/30$ of a loaf. (The Egyptian thought of 2/3 as $1/1\frac{1}{2}$.) Number 33 is: "A quantity whose two-thirds, half and seventh are added to it becomes 37; [what is the number?]" The result is found to be $16 + 1/56 + 1/679 + 1/776$. Professor Peet observes (page 20) "that the problems Nos. 24–38 involve the solution of equations of the first degree with one unknown by means of a simple method of trial," but this statement implies more than he seems prepared to grant in his extended comments on page 60 and those which follow. Only arithmetic is required to solve these problems and if we use our modern arithmetic processes we could often abbreviate those of the papyrus. Surely *this* is what should be emphasized in this connection, when attempting to reproduce the thought of the Egyptians. It is a considerable step in advance to assert that the Egyptians of the twelfth dynasty had any such thought as we have when formulating and solving algebraic equations of the first degree.

Problems 41–60 are problems in mensuration, and include: (*a*) volumes of cylindrical and rectangular parallelopipedal containers; (*b*) expressions for the areas of rectangular, circular, triangular pieces of land; (*c*) the angle of slopes of pyramids.[1] In problem 41 we have the area of a circle of diameter d given as

[1] This is the "seq-et" of Heath's discussion in the part of a paragraph at the top of page 128, volume 1, of his *History of Greek Mathematics*. Oxford, 1921. If space permitted, it might be shown that this part of a paragraph is replete with error and false suggestions. This is probably due to Heath accepting Borchardt and Griffith as authoritative in this connection.

$(8/9)^2d^2$; whence we may find $\pi = 256/81 = 3.1605 -$, instead of the correct value, $3.1416 -$.

Problems 61–84 are miscellaneous arithmetic problems, such as: (*a*) Proportionate values of precious metals—62; (*b*) Division of barley into shares in arithmetical progression—64; (*c*) Daily portion of a yearly ration of fat—66; (*d*) Problem 79, which is apparently the following: "Seven houses; in each are 7 cats; each cat kills 7 mice; each mouse would have eaten 7 ears (or grains) of spelt; each ear of spelt will produce 7 *hekat*. What is the total of all of these?" The result, 7×2801, is (as Peet and Cantor have pointed out) exactly what would be obtained by substituting in the modern formula for the sum, S_n, of n terms of a geometric series $(a = r)$ $S_n = a(a^n - 1)/(a - 1)$. On recalling that $S_n = a(S_{n-1} + 1)$ we note that in the problem before us $a = 7$, $S_{n-1} + 1 = 2801$. Was such a formula, even for simple cases, known to the Egyptians?[1] There is no actual evidence that it was.

Number 85 is an unintelligible group of signs; No. 86 is a fragment of accounts; and No. 87 contains calendrical entries. With this the papyrus ends.

These problems of the Rhind papyrus suggest that the mathematics of Egypt was intensely practical, and Peet believes that this is characteristic of all of the sciences of this country.[2] He remarks: "As Plato alone of the Greeks seems to have realized, the Egyptians were essentially a 'nation of shopkeepers,' and interest in or speculation concerning a subject for its own sake was totally foreign to their minds." If, however, it turns out that the frustum problem of the Moscow papyrus is correctly interpreted, we have a result which appears to indicate that the Egyptians studied mathematics for its own sake. Such problems as those in the Rhind papyrus involving numbers in arithmetic and geometric progressions seem also to imply considerations not strictly practical.

Professor Peet's work is admirably printed, with very clear type and generous spacing and margins, and is attractively and substantially bound. It should be in every college and university library of the country, and no teacher of the early history of mathematics should fail carefully to peruse its pages. The work is one which would also prove of great interest to our host of teachers in the secondary schools. To all such, and to students intellectually curious as to ancient cultures, it is most heartily recommended. May the University of Liverpool be rewarded for publishing this outstanding contribution of English scholarship!

R. C. ARCHIBALD.

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[1] For this suggestion I am indebted to Chancellor A. B. Chace of Brown University.

[2] On the evidence of the Edwin Smith Medical Papyrus alone, J. H. Breasted expressed the belief that the evidence of Egyptian "interest in pure science, as such, is perfectly conclusive." Compare *The New York Historical Society Quarterly Bulletin*, vol. 6, April, 1922, p. 29.

Methodik des mathematischen Unterrichts. By W. LIETZMANN. Second edition, second part. Leipzig, Quelle und Meyer, 1923. Cloth, 8 vo. xii + 367 pages.

It speaks well for the series of handbooks edited by Dr. J. Norrenberg that this work by Dr. Lietzmann, which first appeared in 1916, has met with such success as to warrant a second edition. There is no one in Germany who stands higher than the author as a leader in the reform of the teaching of mathematics. His contributions to the work of the International Commission are well known and the assistance which he rendered to Professor Klein in that great undertaking was very helpful. He is Oberstudiendirektor of the Oberrealschule at Göttingen, a lecturer in the University, and one of the leaders in the pedagogical Seminar. Still a relatively young man, he has the forward look in matters educational, and his editorial duties on the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* have kept him in touch with all the current reform movements in Germany, the most important of which he has recently summarized in an article in *The Mathematics Teacher.*

Such being the equipment of the author, what kind of book has he felt is needed by the German teacher? It would be difficult to find a clearer picture of the difference between the Teutonic point of view and that which prevails in the schools of education in this country. Dr. Lietzmann arranges the 367 pages at his disposal in nine chapters: (1) computation (Der Rechenunterricht), (2) intuitive geometry (Der propädeutische geometrische Unterricht), (3) plane geometry, (4) solid geometry, (5) modern geometry, (6) introduction to algebra (Arithmetik), (7) algebra, and (8) analysis, including (*a*) the function concept, (*b*) analytic geometry, (*c*) the calculus. From this list of topics it is easily seen that the work seeks to present a modern view of the subject matter of instruction, that it attempts no unnatural fusion of topics, and that the author is not one whose mental vision is shut in by the drab walls of educational statistics. This leader in the reform of mathematical teaching has not a word to say about modern tests, leaving this to the trained tester. He would consider as trivial such exercises for teachers as the counting of the problems in factoring in the six "best sellers." What he seeks is to have teachers know the mathematics to be taught and the paths which current and past experience has shown to be the best for their progress. In geometry, for example, he explains and discusses with scientific openness of mind the reforms of Méray and Bourlet and the school which they represented; he considers how much of modern geometry may properly have place in the secondary school; and he takes up such topics as the cinematograph material, the models which have recently appeared, and the modern spirit that is permeating the schools of Europe.

We have nothing of this kind in our American educational literature, and we need it. It is valuable to our teachers to know the nature of the modern tests in mathematics and to be able to use them intelligently, all of which can be taught in a very short time. It is not their business to pose as experts in the making of such tests, however, and it is far more important that they know the subject that they are teaching and the works of real scholarship relating thereto. For

teachers who feel this need and who have some mastery of German this book will be of great help. By others, the book will not be read and will very likely be called reactionary.

D. E. SMITH.

Mensuration and Solid Geometry. By R. M. MILNE. Cambridge University Press, 1923. x + 206 pages, 177 figures.

It is the purpose of this book to develop the formulas most frequently used in mensuration and to afford practice in using them. The author is an instructor in the Royal Naval College at Dartmouth, Eng., and the text is developed with particular regard to the needs of students in that institution but should prove useful to any student who has had training in plane and solid geometry and trigonometry. The style is rapid and condensed. The proofs given are usually brief and frequently informal. The subject matter is varied and well chosen.

No mention is made of logarithms. Not enough emphasis is laid on the determination of the accuracy that may be expected from computations on given data. Computational methods for finding a result to a prescribed number of significant figures are also neglected. The "mathematical" definitions of a cylinder (page 75) and a cone (page 87) are inaccurate.

The collections of problems constitute one of the most commendable parts of the book. The numerous problems of practical value will serve to arouse and maintain the interest of the student. Teachers of geometry and trigonometry who can devote only passing attention to computation will find this book valuable as a source for problems.

C. H. SISAM.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

JOURNAL OF MATHEMATICS AND PHYSICS, Massachusetts Institute of Technology, volume 3, no. 1, January, 1924: "Note on developable surfaces in hyperspace" by C. L. E. Moore, 1–6; "On Ricci's coefficients of rotation" by J. Lipka, 7–23; "Certain notions in potential theory" by N. Wiener, 24–51.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 10, no. 1, January, 1924: "On the subdivision of 3-space by a polyhedron" by J. W. Alexander, 6–8; "An example of a simply connected surface bounding a region which is not simply connected" by J. W. Alexander, 8–10; "Remarks on a point set constructed by Antoine" by J. W. Alexander, 10–12.

QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS, volume 50, no. 1, October, 1923: "Determination of all the characteristic subgroups of an abelian group" by G. A. Miller, 54–61.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 24, no. 4, December, 1922: "Representations of a complex point by pairs of ordered real points" by W. C. Graustein, 245–254; "Non-loxodromic substitutions and groups in n dimensions" by E. B. Van Vleck, 255–273; "Birational transformations simplifying singularities of algebraic curves" by G. A. Bliss, 274–285; "A symbolic theory of formal modular covariants" by O. C. Hazlett, 286–311; "On the mean-value theorem corresponding to a given linear homogeneous differential equation" by

G. Polya, 312–324.—volume 25, no. 3, July, 1923: "Discontinuous boundary conditions and the Dirichlet problem" by N. Wiener, 307–314; "A type of differential system containing a parameter" by F. H. Murray, 315–324; "On a remarkable class of entire functions" by J. I. Hutchinson, 325–332; "Note on an ambiguous case of approximation" by D. Jackson, 333–337; "Expansions in terms of solutions of partial differential equations, Second paper: Multiple Birkhoff series" by C. C. Camp, 338–342; "Circular plates of constant or variable thickness" by C. A. Garabedian, 343–398; "Permutable rational functions" by J. F. Ritt, 399–448; "On approximation by functions of given continuity" by D. Jackson, 449–458.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3076. Proposed by S. A. COREY, Des Moines, Iowa.

Professor Swift has shown in the MONTHLY (*1920*, 301) that the equation,

$$\frac{1}{n+1} - \frac{1}{6} \cdot \frac{4}{2^n} - \frac{1}{6} = 0,$$

must hold in the development of an analytic function, $\int_0^x f(x)dx$, by Simpson's Formula, and that n has only the positive integral values: 1, 2, and 3. The editor (*1920*, 464) has shown that if n be assumed to be a complex number the function, $\int_0^x x^2 \sin(b \log x - c)dx$, is also developable by Simpson's Formula for certain values of b.

In Weddle's Formula,

$$\int_0^h f(x)dx = \frac{h}{20}\left[f(0) + 5f\left(\frac{h}{6}\right) + f\left(\frac{h}{3}\right) + 6f\left(\frac{h}{2}\right) + f\left(\frac{2h}{3}\right) + 5f\left(\frac{5h}{6}\right) + f(h)\right],$$

the equation for finding n may easily be shown to be

$$(1+n)(2^n + 4^n + 6^n + 5^{n+1} + 6.3^n + 5) = 20.6^n,$$

which holds for all positive integral values of $n > 5$.

By taking n to be a complex number find other functions developable by Weddle's Formula, or show that none exist.

3077. Proposed by R. M. MATHEWS, Wesleyan University.

What geometric relations between a curve and its evolute are implied by a point of intersection of the two curves?

3078. Proposed by N. ALTSHILLER-COURT, University of Oklahoma.

Find the curve having the property that the tangent and the normal at any point determine on a fixed line two points conjugate in a given involution.

3079. Proposed by J. J. QUINN, Pittsburgh, Pa.

The line OP is equal to and coincident with the straight line segment AB. As O describes AB uniformly, OP rotates uniformly through 180°. Determine (*a*) the locus of P; (*b*) the length of the curve for a complete revolution; (*c*) the area of a loop.

3080. Proposed by C. N. SCHMALL, New York City.

Given a cyclic quadrilateral $ABCD$ such that its diagonals AC, BD intersect at right angles in P. If O be the center of the circle, and perpendiculars be dropped on the sides of the quadrilateral from O and P, show that the eight feet of these perpendiculars are concyclic.

NOTE: The circle on which these points lie is somewhat analogous to the *nine-point* circle of a triangle. Its center is the middle point of OP.

3081. Proposed by HARRY LANGMAN, New York City.

Show that

$$\begin{vmatrix} \binom{2}{1} & \binom{4}{2} & \binom{6}{3} & \cdots & \binom{2m-2}{m-1} & \binom{2m}{m} \\ 1 & \binom{4}{1} & \binom{6}{2} & \cdots & \binom{2m-2}{m-2} & \binom{2m}{m-1} \\ 0 & 1 & \binom{6}{1} & \cdots & \binom{2m-2}{m-3} & \binom{2m}{m-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{2m}{1} \end{vmatrix} = 2,$$

where the parentheses represent binomial coefficients.

SOLUTIONS.

Solutions of **2980** (*1922*, 271) were received from R. W. GENESE and R. F. DAVIS, Aberystwyth, Wales, and solutions of **2993** (*1922*, 420) and **3002** (*1923*, 41) were received from E. P. BUGDANOFF, Manchuria, China, after solutions of these problems had been sent to the printers.

3005 [1923, 76]. Proposed by F. D. MURNAGHAN, Johns Hopkins University.

Given the recurrence formulæ: $x_{n+1} = x_n(x_n + y_n)$, $y_{n+1} = y_n(2x_n + y_n)$, with the initial values $x_1 = 1$, $y_1 = 1$, it is desired to find a good approximation to $x_n/(2x_n + y_n)$ for large values of n. In particular, for values of n from $n = 20$ to $n = 30$.

Note.—This question arose in connection with a problem in genetics.

SOLUTION BY A. A. BENNETT, University of Texas.

1. The problem may be solved without recourse to any asymptotic formula by a simple transformation. The transformed variable is chosen so as to satisfy a simpler recursion relation. Let $s_n = (y_n + x_n)/x_n$. It is readily verified that the given recursion formulæ for x_n and y_n imply $s_{n+1} = s_n + 1 - 1/s_n$. The desired function is of the form $1/(s_n + 1)$. Using this relation, I have computed s_n on a computing machine for n from 1 to 31 inclusive, and have checked the results. To eight decimal figures, the last digit being possibly erroneous, we have

$s_1 = 2.00000000$
$s_2 = 2.50000000$
$s_3 = 3.10000000$
$s_4 = 3.77741936$
$s_5 = 4.51268836$
$s_6 = 5.29109096$
$s_7 = 6.10209402$
$s_8 = 6.93821585$
$s_9 = 7.79408658$
$s_{10} = 8.66578418$
$s_{11} = 9.55038781$
$s_{12} = 10.44568002$
$s_{13} = 11.34994667$
$s_{14} = 12.26184053$
$s_{15} = 13.18028671$
$s_{16} = 14.10441583$
$s_{17} = 15.03351606$
$s_{18} = 15.96699802$
$s_{19} = 16.90436884$
$s_{20} = 17.84521254$
$s_{21} = 18.78917510$
$s_{22} = 19.73595297$
$s_{23} = 20.68528402$
$s_{24} = 21.63694047$
$s_{25} = 22.59072321$
$s_{26} = 23.54645725$
$s_{27} = 24.50398801$
$s_{28} = 25.46317833$
$s_{29} = 26.42390593$
$s_{30} = 27.38606141$

$s_{31} = 28.34954648.$

This suffices for the practical problem raised.

2. To secure an asymptotic formula by the method of undetermined coefficients presents difficulties, since the general form of s_n as a function of n is not obvious.

Using the functional notation $s(n)$ in place of s_n we have $s(n+1) - s(n) = 1 - 1/s(n)$. If the relation had been $\lim_{h\to 0}\left[\frac{s(n+h)-s(n)}{h}\right] = 1 - \frac{1}{s(n)}$, this could have been written $\frac{ds(n)}{dn} = 1 - \frac{1}{s(n)}$, or $\frac{sds}{s-1} = dn$, whence $ds + \frac{ds}{s-1} = dn$ whence $s + \log(s-1) - c = n$. This relation is not of course the actual solution of the proposed problem and was obtained by virtue of an unjustified alteration. However as frequently occurs in the theory of difference equations a hint as to the nature of the solution of a given problem in differences is often obtained by considering the more familiar differential equation which results upon replacing, when possible, $s(n+1) - s(n)$ by $\lim_{h\to 0}\left[\frac{s(n+h)-s(n)}{h}\right]$, that is, by ds/dn. In the present case the analogous differential equation suggests that it might be much more difficult to solve for s as a function of n than to solve for n as a function of s and $\log(s-1)$.

We therefore write, in this case,

$$n + c = s + \log(s-1) + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3} + \frac{a_4}{s^4} + \cdots,$$

where s stands for s_n. Whence, putting $n+1$ for n,

$$n + 1 + c = s_{n+1} + \log(s_{n+1} - 1) + \frac{a_1}{s_{n+1}} + \frac{a_2}{{s_{n+1}}^2} + \frac{a_3}{{s_{n+1}}^3} + \frac{a_4}{{s_{n+1}}^4} + \cdots.$$

Using the relation $s_{n+1} = s_n + 1 - \frac{1}{s_n}$, and writing t for $1/s_n$, we have

$$n + 1 + c = s + 1 - t + \log(s-t) + a_1\frac{t}{1+t-t^2} + a_2\left(\frac{t}{1+t-t^2}\right)^2 + \cdots.$$

By subtracting the initial expansion from this, and noting that

$$\log\left(s - \frac{1}{s}\right) - \log(s-1) = \log\left(1 + \frac{1}{s}\right) = \log(1+t),$$

we have

$$0 = -t + \log(1+t) + a_1\left(\frac{t}{1+t-t^2} - t\right) + a_2\left[\left(\frac{t}{1+t-t^2}\right)^2 - t^2\right] + \cdots.$$

Now $\frac{t}{1+t-t^2} - t$, when expanded in powers of t, becomes $-t^2 + 2t^3 - 3t^4 + 5t^5 - 8t^6 + 13t^7 \cdots$, the coefficients forming a Farey sequence (with initial unity omitted). Similarly $\left(\frac{t}{1+t-t^2}\right)^2 - t^2$ yields $-2t^3 + 5t^4 - 10t^5 + 20t^6 - 38t^7 \cdots$. From $\left(\frac{t}{1+t-t^2}\right)^3 - t^3$, we obtain $-3t^4 + 9t^5 - 22t^6 + 51t^7 + \cdots$. From $\left(\frac{t}{1+t-t^2}\right)^4 - t^4$ we obtain $-4t^5 + 14t^6 - 40t^7 \cdots$. From the next term we have $-5t^6 + 20t^7 - \cdots$. From the next, $-6t^7 + \cdots$. The expansion of $\log(1+t)$ in powers of t is $t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \frac{t^6}{6} + \frac{t^7}{7} - \cdots$. We have then, altogether, the following equation:

$$0 = (-\tfrac{1}{2} - a_1)t^2 + (\tfrac{1}{3} + 2a_1 - 2a_2)t^3 + (-\tfrac{1}{4} - 3a_1 + 5a_2 - 3a_3)t^4 + (\tfrac{1}{5} + 5a_1 - 10a_2 + 9a_3 - 4a_4)t^5 + (-\tfrac{1}{6} - 8a_1 + 20a_2 - 22a_3 + 14a_4 - 5a_5)t^6 + (\tfrac{1}{7} + 13a_1 - 38a_2 + 51a_3 - 40a_4 + 20a_5 - 6a_6)t^7 + \cdots.$$

Equating the coefficients of the several powers of t to zero, we have, at once, by successive solutions and substitutions,

$$a_1 = -\tfrac{1}{2}, \quad a_2 = -\tfrac{1}{3}, \quad a_3 = -\frac{5}{36}, \quad a_4 = -\frac{13}{240},$$

$$a_5 = -\frac{193}{1800}, \quad a_6 = -\frac{947}{7560}, \quad \cdots.$$

From the character of the computation, it is clear that further coefficients could have been obtained in the same manner, although with increasing difficulty. This use of undetermined

coefficients does not of itself throw any light upon the question of convergence. When these values that have been found are placed in the initial series, we have, as desired,

$$n + c = s + \log(s-1) - \frac{1}{2s} - \frac{1}{3s^2} - \frac{5}{36s^3} - \frac{13}{240s^4} - \frac{193}{1800s^5} - \frac{947}{7560s^6} - \cdots,$$

where s stands for s_n. The constant, c, depends upon the initial values of s_n. To determine c, the values of $n = 15$, $n = 20$, $n = 25$, $n = 30$, $n = 31$ were tried with uniform agreement up to the last digit. Thus c was determined as .64018855.

To test the availability of this asymptotic form for large values of n, s_{100} was computed. This involved the independent computation of $\log s$ for values in the neighborhood of $s = 96$. Despite this added work and the fact that repeated interpolation was necessary, the computation was completed by use of a computing machine within a few minutes. The result obtained was

$$s_{100} = 96.09059614.$$

The use of this asymptotic formula implies that s_n is to be regarded as a continuous real function of n capable of asymptotic representation. No inquiry as to convergence was undertaken, but the formula above agreed most convincingly with the results secured by the recursion formula.

3008 [1923, 76]. Proposed by P. R. RIDER, Washington University.

The altitude of a right circular cone is a, the radius of its base is b, and its slant height is c. A string is wrapped n times about the cone, starting at the vertex and ending at the base, in such a manner that for any complete circuit the vertical rise (the cone being supposed to rest on its base) is the same. A bird at the vertex takes the end of the string in its beak and flies around the cone, unwinding the string, keeping it taut and always tangent to the curve of the string as it lies around the cone. Find an expression for the distance that the bird has flown when the string is completely unwound, (*a*) if it starts at the vertex, (*b*) if it starts at the base.

SOLUTION BY THE PROPOSER.

(*a*) Consider a moving horizontal line which always passes through the axis of the cone and through a point of the string as it lies wrapped around the cone. Choose the limiting position of this line, as it approaches the vertex of the cone, as the x-axis; choose the axis of the cone as the z-axis, and take the plus direction upward; choose the y-axis perpendicular to the x- and z-axes. The origin will of course be the vertex of the cone.

Let θ be the angle through which the horizontal line has rotated for any given position, and denote by r the segment of the line included between the axis of the cone and the string. Then $r = b\theta/2n\pi$, and if x, y, z are the coördinates of a point on the string,

$$x = r\cos\theta = \frac{b\theta\cos\theta}{2n\pi}, \qquad y = \frac{b\theta\sin\theta}{2n\pi}, \qquad z = -\frac{a\theta}{2n\pi}.$$

It can easily be shown that the direction cosines of the tangent line to the curve are

$$\cos\alpha = \frac{b(\cos\theta - \theta\sin\theta)}{\sqrt{b^2\theta^2 + c^2}}, \qquad \cos\beta = \frac{b(\sin\theta + \theta\cos\theta)}{\sqrt{b^2\theta^2 + c^2}}, \qquad \cos\gamma = \frac{-a}{\sqrt{b^2\theta^2 + c^2}}.$$

If s represents the length of the string from the vertex of the cone to the point (x, y, z), then

$$s = \int_0^\theta \sqrt{x'^2 + y'^2 + z'^2}\,d\theta = \frac{1}{4bn\pi}\left(b\theta\sqrt{b^2\theta^2 + c^2} + c^2\log\frac{b\theta + \sqrt{b^2\theta^2 + c^2}}{c}\right). \qquad (1)$$

Denote by X, Y, Z the coördinates of the point on the path of the bird corresponding to the point (x, y, z) on the cone. Then, if we note that the unwound portion of the string extends in the opposite direction from the positive tangent to the curve, we get

$$X = x - s\cos\alpha, \qquad Y = y - s\cos\beta, \qquad Z = z - s\cos\gamma.$$

Differentiation gives

$$\begin{aligned} X' &= x' - s'\cos\alpha - s(\cos\alpha)' = -s(\cos\alpha)', \\ Y' &= y' - s'\cos\beta - s(\cos\beta)' = -s(\cos\beta)', \\ Z' &= z' - s'\cos\gamma - s(\cos\gamma)' = -s(\cos\gamma)'. \end{aligned}$$

The length of the path of the bird is

$$\int_0^{2n\pi} \sqrt{X'^2 + Y'^2 + Z'^2}\,d\theta = b\int_0^{2n\pi} \frac{s}{b^2\theta^2 + c^2}\sqrt{(\theta^2+4)(b^2\theta^2+c^2) - b^2\theta^2}\,d\theta,$$

in which s is given by (1).

(b) Taking the origin at the center of the base, we have, for suitably chosen axes,

$$r = b(1 - \theta/2n\pi),$$

and

$$x = r\cos\theta = b\left(1 - \frac{\theta}{2n\pi}\right)\cos\theta, \qquad y = r\sin\theta = b\left(1 - \frac{\theta}{2n\pi}\right)\sin\theta, \qquad z = \frac{a\theta}{2n\pi}.$$

The direction cosines of the tangent line to the curve are

$$\cos\alpha = \frac{-2bn\pi\sin\theta - b(\cos\theta - \theta\sin\theta)}{\sqrt{b^2\phi^2 + c^2}}, \qquad \cos\beta = \frac{2bn\pi\cos\theta - b(\sin\theta + \theta\cos\theta)}{\sqrt{b^2\phi^2 + c^2}},$$

$$\cos\gamma = \frac{a}{\sqrt{b^2\phi^2 + c^2}},$$

where $\phi = \theta - 2n\pi$.

The length of the string from the end to the point (x, y, z) is found to be

$$s = \frac{1}{4bn\pi}\left[b\phi\sqrt{b^2\phi^2 + c^2} + 2bn\pi\sqrt{4b^2n^2\pi^2 + c^2} + c^2\log\frac{b\phi + \sqrt{b^2\phi^2 + c^2}}{-2bn\pi + \sqrt{4b^2n^2\pi^2 + c^2}}\right]. \qquad (2)$$

If X, Y, Z are the coördinates of the point on the path of the bird corresponding to the point (x, y, z) on the cone, we have

$$X = x - s\cos\alpha, \qquad Y = y - s\cos\beta, \qquad Z = z - s\cos\gamma,$$

and

$$X' = -s(\cos\alpha)', \qquad Y' = -s(\cos\beta)', \qquad Z' = -s(\cos\gamma)'.$$

The distance that the bird flies is found to be

$$b\int_{-2n\pi}^{0} \frac{s}{b^2\phi^2 + c^2}\sqrt{(\phi^2+4)(b^2\phi^2+c^2) - b^2\phi^2}\,d\phi,$$

where s is defined by (2).

3011 [1923, 146]. Proposed by E. T. BELL, University of Washington.

In a certain paper it is stated that "it is easy to prove that, if $p>0$ is an integer, the relation $a_1\sin\frac{\pi}{2p} + a_2\sin\frac{2\pi}{2p} + \cdots + a_{p-1}\sin\frac{(p-1)\pi}{2p} + a_p = 0$ necessitates $a_1 = a_2 = \cdots = a_{p-1} = a_p = 0$, the a's being integers." Prove it.

Remark by A. Pelletier, Montreal, Canada.

If none of the a's are negative, the proposition is evident, since all the sines are positive. If the a's may be any integers, then the proposition is false. For let $p = 3$, then

$$a_1\sin(\pi/6) + a_2\sin(2\pi/6) + a_3 = 0$$

is satisfied by $a_1 = 2, a_2 = 0, a_3 = -1$.

Note by Otto Dunkel, Washington University.

If the a's may be any integers, the proposition is false for an infinite number of cases. For let $p \geqq 3$ be an odd positive integer; then

$$2\sum_{k=1}^{k=(p-1)/2}(-1)^k\sin\left[\frac{(2k-1)\pi}{2p}\right] - (-1)^{(p-1)/2} = 0.$$

Here the coefficients of the sines and the independent term are all integers, and they are not all zero. This includes the special case above.

3013 [1923, 146]. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that $\lim_{m\to\infty} \sum_{r=0}^{r=m-1} \frac{4m}{4m^2+(2r+1)^2} = \frac{\pi}{4}$.

SOLUTION BY ALEXANDER WIENER, Cornell University.

$$\lim_{m\to\infty} \sum_{r=0}^{r=m-1} \frac{4m}{4m^2+(2r+1)^2} = \lim_{m\to\infty} \sum_{r=0}^{r=m-1} \frac{1}{m} \frac{1}{1+\left(\frac{2r+1}{2m}\right)^2}.$$

If the interval from 0 to 1 is divided into m equal parts, the second factor in the sum to the right is the value of $1/(1+x^2)$ at the middle point of the interval from r/m to $(r+1)/m$. Hence by Riemann's definition of the definite integral the limit is

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}.$$

Also solved by the PROPOSER.

3014 [1923, 146]. Proposed by the late T. M. BLAKSLEE, Ames, Iowa.

If 1, p, d are the radius and sides of the regular inscribed pentagon and decagon, and if a triangle be formed from these three lengths, determine the angle opposite p without the use of trigonometric tables.

SOLUTION BY A. M. HARDING, University of Arkansas.

We have $2d \cos A = 1 + d^2 - p^2$, where $d = 2 \sin 18°$, $p = 2 \sin 36°$, and A denotes the angle opposite p.

Write x in place of 18°, then

$$\begin{aligned} 4 \sin x \cos A &= 1 + 4 \sin^2 x - 4 \sin^2 2x \\ &= 1 - 4 \sin x \sin 3x. \end{aligned}$$

Multiply both sides by $\cos x$, then

$$2 \sin 2x \cos A = \cos x - 2 \sin 2x \sin 3x = \cos 5x = \cos 90° = 0.$$

Hence $A = 90°$.

Also solved by THEODORE BENNETT, H. C. BRADLEY, E. P. BUGDANOFF, W. H. HILL, E. J. OGLESBY, A. PELLETIER, J. B. REYNOLDS, and the PROPOSER.

3017 [1923, 205]. Proposed by F. W. PERKINS, JR., Cambridge, Mass.

M. Edouard Lucas, in *Récréations Mathématiques* (vol. 1, pp. 41–51), gives a proof of the following theorem: Any labyrinth can be traversed in such a way as to cover each path twice, and only twice, returning finally to the starting point (provided merely that one has some means of marking paths as they are traversed) by following the three rules given below:

(1) On arriving at a new vertex (*i.e.*, one not visited before), leave by a new path if possible; otherwise return by the same path.

(2) On arriving at an old vertex by a new path, return by the same path.

(3) On arriving at an old vertex by an old path, leave by a new path if possible; otherwise by a path marked just once. It can be shown that it is always possible to do this until every path of the labyrinth has been marked twice.

Assuming Lucas' results, show that, if his rules are observed, the two trips on each path are in opposite directions.

NOTE: We may represent any labyrinth geometrically by a collection of points (" vertices "), not necessarily in a plane, which are joined by plane or twisted curves (" paths ") in any manner whatever, provided merely that it is possible to pass from any point to any other along the curves and that the curves do not meet except at vertices.

Solution by the Proposer.

Let us first consider the particular case that the labyrinth is such that it is possible to go from any given vertex to any other by only one route (*i.e.*, the case that the labyrinth is a "tree"). Suppose some particular path p is traversed the first time from the end P to the end P'. Let us now assume that the second time it is traversed in the same direction. In the intervening part of the itinerary, then, we go from P' to P without going along p. There are, therefore, two routes in the labyrinth, connecting P and P'. But this is impossible, since the labyrinth is a tree. This contradiction establishes the desired result for the case that the labyrinth is a tree.

Let us now consider the general case. We note from rules No. 1 and No. 3 that on arriving at an old vertex by a new path we return by the same path, just as if the vertex were a new vertex from which no other path led. Consider an arbitrary itinerary of the labyrinth in accordance with the rules. Let us imagine that all new paths by which we come to old vertices (on this particular itinerary) have their ends at that vertex blocked, so that the ends of these paths may be regarded as new vertices to which, in each case, only one path leads. This will not interfere with the particular itinerary under consideration, which is in accordance with the rules as applied to either the given labyrinth, or to the modified labyrinth. If the modified labyrinth is a tree, then in the given itinerary each path is traversed once in each direction. Hence, our theorem will be proved if we can show that the modified labyrinth is always a tree.

Suppose that this is not always the case. Then the modified labyrinth corresponding to some itinerary of the given labyrinth must have the property that it has two distinct vertices, P, Q, such that it is possible to go from P to Q along either of two routes, r, r', which are, in part at least, distinct. Consider the different stages of the itinerary at which we make our first trips on the various paths of r, r'. Let p be the last one of these paths on which we make the initial trip. Then both ends of p are old vertices when we make this trip; hence, after traversing p, we arrive at an old vertex by a new path.

From the manner of construction of the modified labyrinth, it is clear, however, that in the itinerary in question, we never come to an old vertex by a new path. This contradiction establishes the fact that every possible modified labyrinth is a tree, and so completes the proof of the theorem.

Note by the Editors:—The proof may also be put in two other forms:

(*a*) If any vertex has only one path ending in it, that path must necessarily be described twice in opposite directions. This part of any itinerary may be obliterated without modifying the application of the rules for the rest. Also, if there is a return to an old vertex by a new path, this path must also be described twice in opposite directions and it may be obliterated. If in any given itinerary this process of obliteration be continued, it may be shown that the whole itinerary is obliterated.

(*b*) Lucas has pointed out that before the completed process any vertex, not the initial vertex, has at a moment before or after entering it two or no paths used only once. It may be easily shown that the initial vertex cannot have any path described twice in the same direction. Now by aid of the fact pointed out by Lucas, it may be shown that there can be no first instance of any vertex having a path described twice in the same direction.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

The following additional announcements of Summer Session courses have been received.

University of Wisconsin, June 30–August 8. By Professor E. B. Skinner: Differential geometry; Theory of algebraic numbers. By Professor Arnold Dresden: Analytic projective geometry; higher algebra. By Professor H. W. March: Mathematical formulation of scientific problems; Fourier series. By Professor W. W. Hart: The teaching of secondary mathematics. By Professor

WARREN WEAVER: Differential equations; Definite integrals; Generalized coordinates. By Mr. E. B. MILLER: Theory of equations. Mathematical Research will be directed by Professors SKINNER, DRESDEN, MARCH, and WEAVER.

Johns Hopkins University, July 7 to August 15. In addition to courses in Trigonometry, Analytic geometry, and Calculus, the following course is offered: By Professor F. D. MURNAGHAN: Differential geometry of curves and surfaces.

Between February 28 and March 11, Professor H. E. SLAUGHT made a tour through the states of Louisiana, Mississippi, Alabama and Georgia speaking on behalf of the interests of mathematics as a whole and, in particular, on behalf of the American Mathematical Society and the Mathematical Association of America. He delivered eight addresses in all, two of them before groups of teachers of collegiate mathematics and their guests from numerous departments related to this field, and six before large groups of students interested in mathematics.

At the University of Georgia, he was the guest of the Southeastern Section of the Association on the occasion of its third annual meeting. At the University of Louisiana he was the official representative of the Association at a conference of Louisiana and Mississippi teachers of collegiate mathematics called for the purpose of organizing a Section in those states. This conference was highly successful and resulted in the drafting of a petition for a charter signed by twenty-nine members and applicants for membership. A program of mathematical papers was carried through after the business session and much interest and enthusiasm were in evidence as surety for the success of Section number fourteen in case a charter is granted. The petition will be acted upon by the Trustees at an early date.

At Tulane University, Professor Slaught was the guest of Professor H. E. Buchanan. He found there an active mathematical club consisting of the faculty of Tulane and Sophie Newcomb, their advanced students majoring in mathematics, and high school teachers of mathematics in New Orleans. A good indication of the vitality of this club was shown by the fact that on a Mardi Gras holiday more than one hundred members turned out to hear an address on the "Romance of the Number System."

At the University of Alabama Professor Slaught was the guest of Professor Tomlinson Fort. He found two active mathematical clubs, one of freshmen and one of Seniors, the latter a chapter of Pi Mu Epsilon. The standing and influence of these clubs may be inferred from the fact that on their invitation nearly three hundred students came to hear this address on the Number System. The impression was strongly corroborated that a great field of usefulness is open to the undergraduate mathematical clubs and that every encouragement should be given to the organization of such clubs where they do not now exist.

Mr. C. C. CRAIG, instructor in mathematics at the University of Michigan, is one of 16 Americans to be granted fellowships for 1924–1925 by the American-Scandinavian Society. He plans to study mathematical statistics under Pro-

fessors C. V. L. CHARLIER and S. D. WICKSELL at the University of Lund, Sweden.

The following appointments and promotions in the department of mathematics at Hunter College of the City of New York have been announced: to succeed Professor EMMA M. REQUA, retired, as head of the department, Professor TOMLINSON FORT of the University of Alabama; to be an associate professor, Miss LAO G. SIMONS who has been acting head of the department for the present year; to be an instructor, Miss HELEN KUNTE.

W. E. BYRNE, who has studied at Rensselaer Polytechnic Institute and at the University of Paris, has been awarded an American Field Service Fellowship for French Universities for the year 1924–1925. Applications for the year 1925–1926 should reach the secretary, at 525 West 120th Street, New York City, not later than December 15, 1924.

At Harvard University, Professor G. D. BIRKHOFF will be exchange professor at Pomona, Colorado, and Grinnell Colleges during the first half of the academic year 1924–25. For the year 1924–25, Mr. H. W. BRINKMANN and Mr. J. L. HOLLEY have been appointed Benjamin Peirce instructors, and Mr. P. D. EDWARDS and Mr. MALCOLM MACLAREN, JR. instructors. Dr. J. L. WALSH has been promoted to an assistant professorship of mathematics.

Professor E. R. HEDRICK, of the University of Missouri, has been appointed professor of mathematics at the University of California and will be in charge of the department of mathematics at the southern branch, Los Angeles.

Professor A. N. WHITEHEAD, of the Imperial College of Science and Technology, London, has been appointed professor of philosophy at Harvard University.

Associate Professor CLARA E. SMITH, of Wellesley College, has been promoted to a full professorship of mathematics.

Dr. H. T. DAVIS, of the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Indiana.

Dr. G. A. PFEIFFER, of Columbia University, has been promoted to an assistant professorship of mathematics.

Dr. GEORGE RUTLEDGE, of Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

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